PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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ATMOSPHERIC WAVES CAUSED BY LARGE EXPLOSIONS

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CONTENTS

This paper considers the harmonic oscillations of several simple model atmospheres. The oscillations are of two types. In the first, the kinetic energy per unit volume tends to zero at great heights; in the second, the kinetic energy per unit volume remains finite. A large explosion at ground level excites a spectrum of both types of oscillation. The pulse ultimately separates into two parts—a train of travelling waves which can be observed at ground level at great distances, and a train of travelling waves which disappear into the upper atmosphere.

The complete range of experimental observations on the pressure oscillations caused by explosions of energies varying between 10^{20} and 10^{24} ergs can only be interpreted with model atmospheres having one or more sound channels, i.e. having at least one minimum in the temperature–height relationship of the atmosphere. In spite of the complexity of the phenomena, the theory throws light on some of the characteristic features of the observations. The average period of the largest waves is roughly proportional to the cube root of the energy released by the explosion. The amplitudes of the waves from large explosions can be calculated. Conversely, good records enable the size of the explosion to be estimated.

The energy of the Siberian meteorite of 1908 was about 10^{16} cal, or 10 MT (T signifying a ton of t.n.t.).

1. Introduction

There have been several attempts at a theory of the waves generated in the atmosphere by the release of energy from a source which is concentrated in space and time. The Krakatoa volcanic eruption of 1883, and the great Siberian meteorite of 30 June 1908, are the two natural phenomena to which the theory has so far been applied, but today the air waves generated by a point source of energy have a new important application. Some nuclear explosions in the atmosphere are sufficiently powerful to generate a train of atmospheric waves which can be detected by sensitive but standard microbarographs at distances of

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a few thousand kilometres. For example, Japanese meteorological stations have recorded air waves from nuclear explosions made by the United States and by Russia. Specially developed arrays of microbarographs are capable of recording waves from point sources of energy (nuclear explosions, meteors, rock slides, volcanic eruptions) as small as about 10^{20} ergs at distances exceeding 1000 km. The East–West Conference on the technical feasibilities of a world-wide control system for detecting and identifying nuclear explosions, held at Geneva in July and August of 1958, discussed the 'acoustic' and other methods of detection at great length. However, it was clear that no satisfactory theory of the formation of the 'acoustic' waves has so far been constructed. The present paper is an attempt at clarifying the physics of the situation and giving a basic theoretical interpretation of the freely travelling atmospheric waves, or 'gravity waves' as they are often called.

The early versions of the theory have sought mainly to explain the observation that the speed of propagation of the disturbance is slightly less than the speed of sound at ground level (see Lamb's *Hydrodynamics*).

The later contributions of Pekeris (1937, 1939, 1948) and particularly of Scorer (1950) went further and attempted to derive pressure–time variation in the air at ground level from an explosion at ground level.

There appears to be one route which makes analysis manageable, and this route falls into two parts. The first is a discussion of the harmonic oscillations of various simple model atmospheres in terms of wavelength and frequency; the second is the representation of the wave system generated by an explosion as a Fourier integral. Even an idealized treatment of this type involves a great deal of arithmetic, and some of the numerical results described in the present paper were obtained by the use of a large computer, the I.B.M. 704.

The discussion of the free oscillations of the atmosphere proceeds as follows. The hydrodynamic equations are linearized, assuming axial symmetry and neglecting dissipative terms. Simple models of the atmosphere are assumed. Winds and topographical features such as mountain ranges are neglected. Periodic oscillations are sought. Sometimes the vertical velocity is taken as negligible, in which case the rotation of the earth must be included because the harmonic periods of interest are several hours in duration (see, for example, Pekeris 1937, 1939, 1948; Wilkes & Weekes 1947). In other cases, such as are treated in the present paper, the vertical velocity is retained but the rotation of the earth can be neglected because the harmonic periods of interest are of the order of minutes or seconds.

To represent an explosive source, Fourier integrals of the fundamental modes are constructed; and the evaluation of these integrals at later times gives the wave train generated by the source.

Though discussion of the free oscillations is subject to all of the approximations of linearized theory, and even then involve obscurities such as the velocity increasing indefinitely with height, there is no doubt that the results following from this part of the treatment are physically acceptable in the problems we are attempting to solve. It is otherwise with the second part which treats the explosive source by means of Fourier integrals. The difficulty is to represent the phenomena arising from an explosion with any degree of completeness by means of a Fourier analysis applied to a realistic model of the atmosphere.

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An explosion in the air sends out a blast wave led by a shock which heats the air irreversibly. Near the place where the explosion occurs, the air is left hot and expanded, even when the pressure has returned to atmospheric. Scorer (1950) calculates the gravity waves in the atmosphere as equal to those which would be caused by the introduction of a certain volume of air into the atmosphere. This volume is taken as equal to the expansion of the air caused by heating of the air at constant pressure by the whole energy of the explosion. If the explosion releases energy E cal the expansion is then $11E\,\mathrm{cm}^3$.

In stating these approximate figures, we have glossed over the fact that a great deal of radiant energy is emitted by the shock front in the early time after the explosion has occurred. A nuclear explosion in 'free air' may radiate one-half of the total energy in the form of light and heat. Presumably the major part of this radiant energy escapes 'to infinity' in the sense that it does not contribute to the gravity waves. A little of the energy is absorbed in the air within a few miles of the explosion, and this energy will contribute. Some of the energy will vaporize some of the material of the ground, and because the vapour will recondense within a few seconds, some of this energy will again be put into the air, thus adding a little to the energy available for making gravity waves. A nuclear explosion on the ground will radiate less heat and light 'to infinity' than the same explosion well above the ground. A large chemical explosion on the ground, or a natural event such as the impact of the Siberian meteorite, will radiate less energy as light and heat than a nuclear explosion of the same total energy, but the numerical values are not known. The energy put into the ground as shock or cratering from any large surface explosion is only a few parts per cent of the total energy.

The kinetic energy in the blast wave can be calculated from experimental data given by U.S.A.E.C. (1957). When the blast from a nuclear explosion on the ground has reached a radius at which the shock overpressure is 1 Lb./in.^{2}, the kinetic energy in the positive phase is approximately 8 % of the total energy released by the explosion, and the kinetic energy in the negative phase is approximately 0·5 %. Hence the energy at this stage has been partitioned approximately in the following way: kinetic energy in the blast 10 %, radiated away from the region of the explosion as electromagnetic radiation 20 to 30 %, cratering and ground shock 1 to 5 %, and the remainder of about 55 to 70 % as heat and compressional energy in the air.

Thus, the figure of 11 cm^3 expansion per calorie of explosive energy must be regarded as an upper limit. It might apply to small chemical explosions well above the ground.

The most satisfactory method of estimating the expansion of the air per calorie of explosion energy is to fit the observed pressure-time variation in the blast from explosions, at a low pressure level, to the theory of sound, and thus calculate the point source which would by sound theory cause the blast wave at this low pressure level. This calculation is made in § 7. It appears that a nuclear explosion on the ground causes an expansion of 8 cm^3 per calorie of energy released.

In the case of the Siberian meteorite, we guess that the amount of light and heat radiated was less than it would be from a nuclear explosion. However, more energy would proportionately have gone into the ground. The figure of 8 cm^3 of air expansion per calorie of total energy might again be a reasonable approximation.

The hot air bubble left behind by a large explosion rises rapidly as a nearly spherical vortex, entraining air as it rises, and effectively coming to rest in 5 to 10 min at the 'stabilized height'. We must consider if this rising motion is an efficient producer of gravity waves.

The motion of the air caused by the rise of the hot air bubble is very roughly the same as that in potential flow in an incompressible medium. The bubble from a 1 megaton explosion on the ground is at a height 7 km 1 min after burst. The radius is less than 2 km and the rate of rise is 7×10^3 cm/s. At a horizontal distance of 10 km from the centre of the bubble, the pressure due to the rising motion of the bubble will reach a (negative) maximum of about 80 dyn/cm^2. This pressure pulse would propagate outwards with an amplitude decreasing at least as fast as r^{-1}, and the maximum (negative) pressure at 1000 km

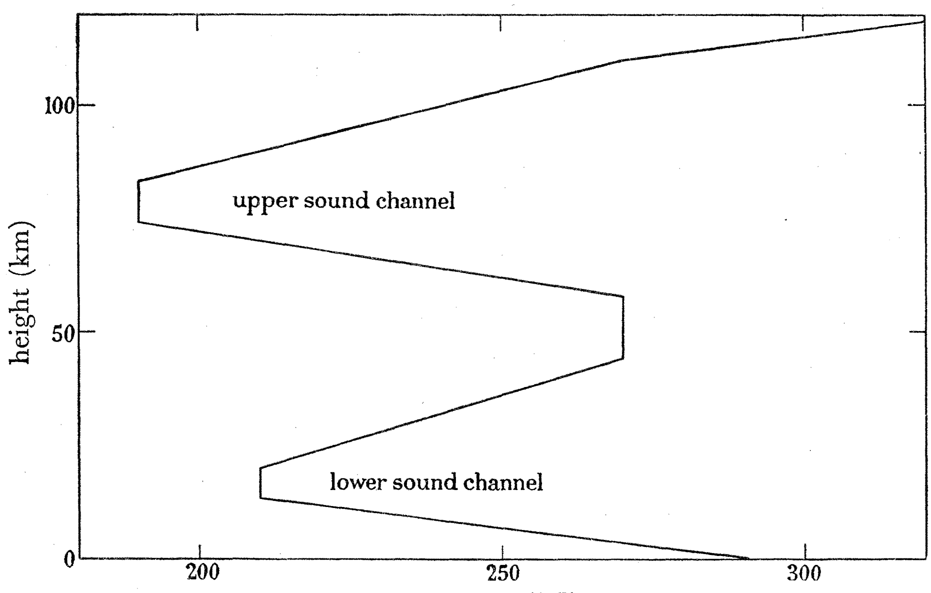

FIGURE 1. The temperature–height variation of the atmosphere based on pressure and density data obtained on rocket flights above White Sands, New Mexico (Havens, Koll & La Gow 1952). There are two well-defined sound channels.

would be at most only a few dynes/cm^2 at ground level. This is two orders of magnitude smaller than the pressure oscillations arising from the blast. Similar conclusions apply to kiloton explosions. We may therefore neglect the gravity waves due to the rising motion of the hot air bubbles. In brief, the great heat content of the hot air bubble causes a mass of air to be lifted, but this motion is an inefficient producer of gravity waves.

In the present paper, we are concerned almost entirely with trains of gravity waves which can be observed at ground level from explosions on or near the ground. Reasons will be given for approximating to the atmosphere with various two-layer models, and more realistically with three-layer models. These models are adequate to give an explanation of the mechanics of the phenomena in fair numerical agreement with experiment.

However, we do not discuss the effects of winds. In comparing our numerical results with observations, we take average experimental values and assume that these are what would be obtained if there were no winds.

Yamamoto (1957) has recently published an interesting paper reproducing some of the microbarograms of nuclear explosions, given in earlier papers, and calculating the normal modes of two-layer and four-layer atmospheres. His mathematics is superficially different from that given in the early parts of the present paper, but fundamentally the two treatments are similar. However, his arithmetic is not accurate enough nor sufficiently complete for our purpose. Yamamoto is interested in the fact that the microbarograms of the pulses from large explosions begin with waves of relatively long period, but the later waves have periods perhaps as small as 0·2 min. A two-layer atmosphere, with the upper layer colder than the lower layer, has a cut-off period of not less than 1·5 min and Yamamoto deduces that the microbarograms cannot be explained in terms of a two-layer model. Yamamoto therefore studied the harmonic oscillations of a four-layer atmosphere, the top layer being the warmest of the four. He finds that harmonic oscillations with a period 0·5 min, and probably shorter, are possible. Moreover, the group velocity is not a monotonic function of period, and has a minimum. We agree with these conclusions, but Yamamoto has missed the existence of additional solutions which are of fundamental importance for understanding the gravity waves caused by large explosions. Yamamoto does not consider the problem of calculating the pulse generated by an explosion.

Since it is necessary, later in this paper, to approximate to the atmosphere with various simplified models, it will be useful to have available the best average temperature–height distribution. This is shown in figure 1.

2. Pressure equations

We follow Scorer (1950) in his derivation of linearized equations for the spatial variation of pressure for a simple harmonic motion in the atmosphere with a given frequency σ. Since dissipative processes are neglected, we consider only real values of σ.

The suffix zero will denote values in a stationary unperturbed atmosphere. Spherical polar co-ordinates (θ, ϕ, z) will be used to denote the point where the velocity is $\mathbf{u}$ or (u, v, w). Great circle distances on the earth's surface are given by $r = a\theta$, where a is the radius of the earth.

The reciprocal of potential temperature is denoted by

$$\tau+\tau_0 = (T+T_0)^{-1}\,[(p+p_0)/p_{01}]^{(\gamma-1)/\gamma} \tag{1}$$

and a 'modified' pressure by

$$\varpi+\varpi_0 = [\gamma R/(\gamma-1)]\,[(p+p_0)/p_{01}]^{(\gamma-1)/\gamma}, \tag{2}$$

where $T+T_0$ is the absolute temperature, $p+p_0$ the pressure, $\rho+\rho_0$ the gas density, R the gas constant and γ the ratio of specific heats. In a horizontally stratified atmosphere, τ_0, ϖ_0, T_0, p_0, ρ_0 are all functions of z alone. The perturbation variables τ, ϖ, T, p, ρ are assumed sufficiently small for terms of the second and higher orders to be neglected. The equation of state is

$$p+p_0 = (\rho+\rho_0)\,R(T+T_0), \tag{3}$$

where R is the gas constant per gram of air ($2{\cdot}87\times10^{6}$ c.g.s.).

Since the motion is assumed adiabatic, we have the relation

$$\mathrm{D}(\tau+\tau_0)/\mathrm{D}t = 0$$

or

$$\mathrm{i}\sigma\tau+\tau_0' w = 0, \tag{4}$$

where $$\mathrm{D}/\mathrm{D}t = \mathbf{u}.\mathrm{grad}+\partial/\partial t$$
and primes denote $\mathrm{d}/\mathrm{d}z$.

If we neglect the earth's rotation, the equations of motion and continuity are

$$(\rho+\rho_0)\,\mathrm{D}\mathbf{u}/\mathrm{D}t = -\mathrm{grad}\,(p+p_0)+(\rho+\rho_0)\,(0,\,0,\,-g), \tag{5}$$

$$\mathrm{D}(\rho+\rho_0)/\mathrm{D}t+(\rho+\rho_0)\,\mathrm{div}\,\mathbf{u} = 0. \tag{6}$$

Writing the adiabatic condition in the form

$$\mathrm{D}(p+p_0)/\mathrm{D}t = c^2\mathrm{D}(\rho+\rho_0)/\mathrm{D}t, \tag{7}$$

where $$c^2 = \gamma p_0/\rho_0 = \gamma RT_0,$$

we can eliminate $(p+p_0)$ and $(\rho+\rho_0)$ between equations (5), (6) and (7) giving

$$-gw+\mathrm{i}\sigma\varpi/\tau_0+c^2\,\mathrm{div}\,\mathbf{u} = 0, \tag{8}$$

where c is a function of z only.

From equations (1), (2) and (3), we find that τ and ϖ are related by

$$(\tau+\tau_0)\,\mathrm{grad}\,(p+p_0) = (\rho+\rho_0)\,\mathrm{grad}\,(\varpi+\varpi_0),$$

so that, for small motions, equation (5) reduces to

$$(\tau+\tau_0)\,\partial\mathbf{u}/\partial t = -\mathrm{grad}\,(\varpi+\varpi_0)+(\tau+\tau_0)\,(0,\,0,\,-g).$$

The three components of this equation are

$$\mathrm{i}a\sigma\tau_0 u+\partial\varpi/\partial\theta = 0, \tag{9}$$

$$\mathrm{i}a\sin\theta\sigma\tau_0 v+\partial\varpi/\partial\phi = 0, \tag{10}$$

$$\mathrm{i}(\sigma^2\tau_0+g\tau_0')\,w+\sigma\,\partial\varpi/\partial z = 0, \tag{11}$$

where we have used the relation between the unperturbed functions, namely

$$g\tau_0+\partial\varpi_0/\partial z = 0.$$

Taking equations (4), (8), (9), (10) and (11), we can eliminate u, v, w and τ, thereby obtaining an equation for the modified pressure ϖ,

$$\sin^2\theta(\cot\theta+\partial/\partial\theta)\,\partial\varpi/\partial\theta+\partial^2\varpi/\partial\phi^2+a^2\sin^2\theta\sigma^2c^{-2}\varpi$$
$$-\tau_0 a^2\sin^2\theta(gc^{-2}-\partial/\partial z)\,[\{\sigma^2/(\sigma^2\tau_0+g\tau_0')\}\,\partial\varpi/\partial z] = 0.$$

This equation is separable, and if we write

$$\varpi = \varpi_1(z)\,\varpi_2(\theta)\,\varpi_3(\phi)\,\mathrm{e}^{\mathrm{i}\sigma t}, \tag{12}$$

then

$$\varpi_1''(z)-[\{(g\tau_0''+\sigma^2\tau_0')/(g\tau_0'+\sigma^2\tau_0)\}+gc^{-2}]\,\varpi_1'(z)+(g\tau_0'/\tau_0+\sigma^2)\,(c^{-2}-k^2\sigma^{-2})\,\varpi_1(z) = 0, \tag{13}$$

$$(\cot\theta+\partial/\partial\theta)\,\varpi_2'(\theta)+(a^2k^2+m^2\,\mathrm{cosec}^2\,\theta)\,\varpi_2(\theta) = 0, \tag{14}$$

and $$\varpi_3''(\phi)-m^2\varpi_3(\phi) = 0, \tag{15}$$

where k^2 and m^2 are separation constants. Clearly $\varpi_1(z)$ is a function also of the parameters σ and k; $\varpi_2(\theta)$ is a function of the parameters k and m, and $\varpi_3(\phi)$ is a function of m. It may be seen that the θ variation in ϖ given by equation (14) is independent of the nature of the unperturbed temperature–height structure of the atmosphere, unlike the z-component ϖ_1.

If we consider only motions in which there is symmetry about the ϕ axis, then $m = 0$ and from equations (9), (10) and (11) we find

$$u = (\mathrm{i}/\sigma\tau_0)\,\mathrm{e}^{\mathrm{i}\sigma t}\,\varpi_1(z)\,\partial\varpi_2(r/a)/\partial r, \tag{16}$$

$$v = 0, \tag{17}$$

$$w = [\mathrm{i}\sigma/(\sigma^2\tau_0+g\tau_0')]\,\mathrm{e}^{\mathrm{i}\sigma t}\,\varpi_2(r/a)\,\partial\varpi_1(z)/\partial z. \tag{18}$$

For this case, the solution of equation (14) is

$$\varpi_2(\theta) = P_\nu^0(\cos\theta), \tag{19}$$

where

$$\nu(\nu+1) = k^2a^2. \tag{20}$$

Since ka is large compared with unity, ka is very nearly $(\nu+\frac{1}{2})$. When r/a is small, the solution $\varpi_2(\theta)$ reduces to $J_0(kr)$.

When r/a is not small, the asymptotic form of (19) is $(r/a\sin\theta)^{\frac{1}{2}}\,J_0(kr)$. Thus, the wave-length at r is $2\pi/k$, the same as it would be on a 'flat-earth' at distance r, but the amplitude is increased because on the sphere the 'perimeter' is only $2\pi a\sin\theta$, rather than $2\pi r$.

The solution given above for ϖ_2 is appropriate for constructing Fourier integrals which represent a source which introduces volume into the atmosphere at ground level by a singularity.

3. Solutions for the vertical variation of pressure

Pekeris (1948) has solved equation (13) for the vertical variation of pressure in an isothermal atmosphere. Pekeris (1948) and Scorer (1950) obtained solutions for an atmosphere consisting of a troposphere having a constant lapse-rate and an isothermal stratosphere with continuity of temperature at the tropopause. This temperature structure we refer to as model A. In the present paper, Scorer's numerical results for this model A atmosphere are corrected in some points and are generalized to several different temperatures. A later section will treat an atmosphere consisting of two isothermal layers, referred to as model B, and it is shown that most of the characteristics of model A may be represented in this way, together with a number of new features associated with warm layers in the upper atmosphere.

The form taken by equation (13) in the two regions of a model A atmosphere are now derived.

Constant lapse-rate

Pekeris and Scorer represented the troposphere by a region in which the lapse-rate is a constant which is somewhat less than the dry adiabatic lapse-rate. Let the unperturbed temperature be given by

$$T_0 = -\mu(\gamma-1)\,gz/\gamma R, \tag{21}$$

where $0 < \mu < 1$, and the origin of z is suitably chosen, so that substituting in equation (1) gives

$$\left.\begin{aligned}\tau_0'/\tau_0 &= (1-\mu)/\mu z,\\ \tau_0''/\tau_0' &= (1-2\mu)/\mu z.\end{aligned}\right\} \tag{22}$$

By means of the transformation

$$\varpi_1(z) = z^{-\frac{1}{2}m}(z+b)^{\frac{1}{2}}\,\chi(z), \tag{23}$$

Scorer reduced equation (13), using (22), to the form

$$\chi''(z) = S(z)\,\chi(z), \tag{24}$$

which is particularly suitable for numerical integration. Equation (24) has an explicit solution only for $\sigma = 0$ when it is of the form

$$\chi \sim z^{\frac{1}{2}} J_\nu(\beta z^{\frac{1}{2}}),$$

where $\nu^2 = 1+4P, \quad P = \gamma[\gamma(1-2\mu)+2\mu]/4\mu^2(\gamma-1)^2$

and $\beta^2 = \{-4(1-\mu)\,g/\mu\}\lim_{\sigma\to 0}(k/\sigma)^2.$

For any other value of σ, a numerical integration through the troposphere is required subject to the boundary conditions discussed in the next section.

Constant temperature

If a region of the atmosphere is assumed to be isothermal at temperature T_s then

$$\tau_0'/\tau_0 = \tau_0''/\tau_0' = -(\gamma-1)\,g/c_s^2,$$

where $c_s^2 = \gamma R T_s$. Equation (13) then reduces to

$$\varpi_1''(z) - \{(2-\gamma)\,g/c_s^2\}\,\varpi_1'(z) + \{\sigma^2-(\gamma-1)\,g^2c_s^{-2}\}\,(c_s^{-2}-k^2\sigma^{-2})\,\varpi_1(z) = 0. \tag{25}$$

The solution of (25) is

$$\varpi_1 = C\,e^{\kappa z} + D\,e^{\lambda z}, \tag{26}$$

where

$$\left.\begin{aligned} \kappa, \lambda &= (2-\gamma)\,g/2c_s^2 \pm D_s, \\ D_\alpha &= [\{(2-\gamma)\,g/2c_\alpha^2\}^2 + \{(\gamma-1)\,g^2c_\alpha^{-2}-\sigma^2\}\,(c_\alpha^{-2}-k^2\sigma^{-2})]^{\frac{1}{2}}. \end{aligned}\right\} \tag{27}$$

4. Boundary conditions for free oscillations

For free oscillations in a stratified atmosphere, we must impose continuity conditions at each interface together with one boundary condition at the ground and a second for the asymptotic behaviour of the motion in the upper stratosphere. In a non-dissipative model of the atmosphere, a physically acceptable condition for this upper boundary condition is that suggested by Meissner (1921) and used by others, including Pekeris (1937), namely, that the total kinetic energy in a vertical column of air shall be finite for a physically realizable periodic oscillation of the atmosphere. This must be the case if the oscillations are to be generated by a source of finite energy. In an isothermal stratosphere

$$\left.\begin{aligned} \tau_0 &\sim \exp[-(\gamma-1)\,gz/c_s^2], \\ \rho_0 &\sim \exp[-\gamma g z/c_s^2], \end{aligned}\right\} \tag{28}$$

and from the solution and (27) we find

$$\mathbf{u} \sim \exp[\gamma g z/2c_s^2 + D_s z].$$

Thus the kinetic energy density varies as

$$\rho_0 u^2 \sim \exp(\pm 2D_s z). \tag{29}$$

If the quantity under the square root sign of (27) is real and positive, then only the negative sign is permissible in the exponents of (26). These are the oscillations which we use in constructing Fourier integrals to represent the freely travelling gravity waves.

The oscillations which have a finite total energy, however, are not the only oscillations with physical significance. If the quantity under the square root sign of (27) were real and negative, then the kinetic energy per unit volume at great height would be bounded and not tend to zero. The energy in any vertical column would be infinite. However, from oscillations of this type, one can construct upward travelling, dispersing pulses of finite energy content. Taking rectangular co-ordinates (x, y, z), then for example there are periodic oscillations in a completely isothermal atmosphere given by choices from the following

$$\varpi = \begin{matrix}\cos \sigma t \\ \sin \sigma t\end{matrix} \quad \begin{matrix}\cos kx \\ \sin kx\end{matrix} \quad e^{\nu z} \begin{matrix}\cos \beta z \\ \sin \beta z\end{matrix}, \tag{30}$$

where $$\nu = (2-\gamma)\, g/2c^2, \quad \beta^2 = -D_\alpha^2.$$

Considering only harmonic oscillations with a finite energy in any vertical column, we have a solution in the stratosphere

$$\varpi_1(z) = D\, e^{\lambda z}.$$

We now join this to a solution of equations (23) and (24) for a troposphere having a constant lapse-rate. At the tropopause, p and w must be continuous. Eliminating τ between (4) and (11) shows that continuity in w implies continuity in the quantity

$$\varpi_1'/(g\tau_0' + \sigma^2 \tau_0).$$

Together with continuity in ϖ_1, this leads to the condition

$$\frac{\chi_2'}{\chi^2} = \frac{m}{2z_2} - \frac{1}{2(z_2+b)} + \frac{\lambda\{\sigma^2 c_s^2 - (1-\mu)(\gamma-1)\, g^2\}}{\sigma^2 c_s^2 - (\gamma-1)\, g^2}, \tag{31}$$

where $z = z_2$ denotes the tropopause, and use has been made of the transformation (23).

At the ground, denoted by $z = z_1$, we have

$$w = 0$$

and again using (23) this gives the condition

$$\frac{\chi_1'}{\chi_1} = \frac{m}{2z_1} - \frac{1}{2(z_1+b)}. \tag{32}$$

With the equation for $\varpi_1(z)$ in the form (24) and the boundary conditions (31) and (32), Scorer carried out the numerical integration for $\chi(z)$ for a range of values of σ. He verified Pekeris's conclusion that for each value of σ from zero up to a critical value σ_c, there is only one value of k, denoted by k^*, for which $\chi(z)$ satisfies both boundary conditions. For these solutions

$$(k^*/\sigma)^2 < 1/c_s^2$$

and the cut-off frequency is defined by $D_s = 0$.

This integration has been carried out again for the atmospheric conditions assumed by Scorer, and also for a number of other values of ground temperature, height of tropopause and lapse-rate. The solution for $k = k^*(\sigma)$ for these sets of conditions is shown in table 1. It is particularly noticeable that an increase in ground temperature and lapse-rate reduces the cut-off frequency σ_c, and also reduces the group velocity at the cut-off. For mean European conditions chosen by Scorer, the period at the cut-off is approximately 2 min. For mean Pacific conditions, the cut-off is about 2·5 min.

Table 1. Free-wave characteristics for type A atmosphere (c.g.s. units)

atmosphere	I	II	III
surface temperature (°C)	13·7	23·0	33·0
stratosphere temperature (°C)	−43·7	−61·0	−73·0
height of tropopause (m)	9610	12800	15850
sound velocity at surface (m/s)	339·6	345·0	350·8
sound velocity in stratosphere (m/s)	303·7	292·0	283·6

	I		II		III	
$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$
0	3·1895	3·1895	3·2100	3·2100	3·1947	3·1947
1	3·1905	3·1924	3·2131	3·2194	3·1989	3·2076
2	3·1915	3·1955	3·2162	3·2290	3·2034	3·2228
3	3·1925	3·1985	3·2195	3·2391	3·2084	3·2401
4	3·1936	3·2020	3·2228	3·2498	3·2139	3·2597
5	3·1946	3·2053	3·2262	3·2612	3·2199	3·2825
6	3·1957	3·2089	3·2298	3·2733	3·2263	3·3086
7	3·1968	3·2123	3·2334	3·2868	3·2335	3·3381
8	3·1979	3·2161	3·2372	3·3014	3·2413	3·3711
9	3·1990	3·2197	3·2414	3·3174	3·2497	3·4087
10	3·2002	3·2239	3·2458	3·3347	3·2589	3·4528
11	3·2014	3·2278	3·2504	3·3533	3·2691	3·5073
12	3·2027	2·2323	3·2551	3·3736	3·2806	3·5752
13	3·2039	3·2364	3·2601	3·3961	3·2937	3·6536
14	3·2052	3·2411	3·2655	3·4213	3·3084	3·7424
15	3·2065	3·2455	3·2713	3·4496	3·3247	3·8396
16	3·2078	3·2507	3·2774	3·4820	—	—
17	3·2091	3·2555	3·2841	3·5192	—	—
18	3·2105	3·2610	3·2912	3·5627	—	—
19	3·2119	3·2661	3·2991	3·6152	—	—
20	3·2134	3·2721	—	—	—	—
21	3·2147	3·2777	—	—	—	—
22	3·2162	3·2842	—	—	—	—
23	3·2179	3·2902	—	—	—	—
24	3·2196	3·2973	—	—	—	—
25	3·2208	3·3038	—	—	—	—
26	3·2223	3·3116	—	—	—	—
27	3·2241	3·3188	—	—	—	—
28	3·2264	3·3273	—	—	—	—
29	3·2295	3·3370	—	—	—	—
30	—	—	—	—	—	—

Note. An interesting numerical comparison is the value of the reciprocal of the group velocity, $\mathrm{d}k^*/\mathrm{d}\sigma$, and the reciprocal of the average of the sound speed at the surface and at the tropopause. This average is 3·1090 for atmosphere I, 3·1396 for atmosphere II and 3·1526 for atmosphere III, all times 10^{-5} s/cm.

5. An atmosphere consisting of two isothermal layers

In order to simplify the numerical calculations associated with the model A atmosphere used by Pekeris and Scorer, it was thought that many essential features of the problem would become more evident if the model were replaced by another containing two isothermal layers—model B. This appears to be the case, for if the constant-lapse-rate troposphere is replaced by an isothermal layer whose temperature is that obtaining midway between the ground and the tropopause, then the free-wave characteristics $k = k^*(\sigma)$ are found to be very similar to those of model A. Table 2 shows the results obtained in this way, and no doubt the agreement could be improved further by optimizing the temperature in the lower layer instead of simply taking the mean value.

We consider a stratosphere at temperature T_s above a troposphere at temperature T_t. In the stratosphere

$$\varpi_1 = D\,e^{\lambda_s z}, \tag{33}$$

where

$$\lambda_s = (2-\gamma)\,g/2c_s^2 - D_s. \tag{34}$$

In the troposphere

$$\varpi_1 = E(-\lambda_t e^{\kappa_t z} + \kappa_t e^{\lambda_t z}), \tag{35}$$

where

$$\kappa_t, \lambda_t = (2-\gamma)\,g/2c_t^2 \pm D_t. \tag{36}$$

The upper sign in (36) refers to κ_t and the lower to λ_t. It will be seen that (35) automatically satisfies the condition at the ground $z_1 = 0$. As before, we require continuity at the tropopause in

$$\varpi_1 \quad \text{and} \quad \varpi_1'/(g\tau_0' + \sigma^2\tau_0).$$

Table 2. Free-wave characteristics for a type B atmosphere simulating a Scorer type A atmosphere (I of table 1)

troposphere temperature (°C)	−15·0
stratosphere temperature (°C)	−43·7
height of tropopause (m)	9610

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$	$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$
0	3·2007	3·2007	16	3·2146	3·2479
2	3·2022	3·2052	18	3·2167	3·2586
4	3·2038	3·2101	20	3·2190	3·2655
6	3·2054	3·2152	22	3·2214	3·2751
8	3·2070	3·2008	24	3·2239	3·2863
10	3·2088	3·2068	26	3·2266	3·2985
12	3·2106	3·2332	28	3·2295	3·3123
14	3·2126	3·2400	30	3·2325	3·3273

Eliminating the ratio D/E from the resulting two equations, we have after some simplification

$$(1-c_s^2k^2/\sigma^2)\,\{D_t \coth(D_t z_2) - (2-\gamma)\,g/2c_s^2\} + (1-c_t^2k^2/\sigma^2)\,\kappa_s \exp\{(\gamma-1)\,gz_2(c_t^{-2}-c_s^{-2})\} = 0, \tag{37}$$

where

$$\kappa_s = (2-\gamma)\,g/2c_s^2 + D_s. \tag{38}$$

Equation (37) determines $k = k^*(\sigma)$ for assigned values of T_s and T_t or c_s^2 and c_t^2. It illustrates one of the important features of the problem; there are in fact two distinct cases to be considered.

When $T_s < T_t$ we have a cold layer above a warm layer. This model closely represents Scorer's model A atmosphere. The radical D_t becomes imaginary for sufficiently large values of σ, and a cut-off is reached because $(k^*/\sigma)^2 < 1/c_s^2$. Up to this value, D_t remains real, $D_t \coth(D_t z_2)$ is single valued, and only one root for $k = k^*(\sigma)$ is found. This corresponds closely to Scorer's solution as may be seen by comparing table 2 with atmosphere I of table 1.

The limiting case of the isothermal atmosphere $T_s = T_t$ is interesting. There is no point in distinguishing between the stratosphere and the troposphere; and the function ϖ_1, at all heights, is given by (33). The condition that the vertical velocity is zero at the ground usually requires ϖ_1' to be zero at the ground, i.e. $\lambda = 0$ at the ground. There are two values of σ which make λ, as given by (34), zero, namely

$$\sigma = kc \quad \text{and} \quad \sigma^2 = (\gamma-1)\,g^2/c^2.$$

The first root genuinely makes w zero at the ground, and also implies that the vertical velocity is always zero at all heights.

The second root makes ϖ_1' zero at the ground, but it does not make w zero at the ground. Referring to equation (18), it will be seen that the condition ϖ_1' is zero at the ground makes $(\sigma^2\tau_0+g\tau_0')\,w$ zero at the ground; and the second root above has only expressed the fact that there is a value of σ which makes $(\sigma^2\tau_0+g\tau_0')$ zero for an isothermal atmosphere.

Pekeris (1948), working in terms of the divergence of velocity rather than our function $\varpi_1(z)$, obtained a second mode of oscillation for an isothermal atmosphere for which $\sigma^2 = gk$. This solution is fictitious since it makes a multiplicant of w_1, and not w_1 itself, zero.

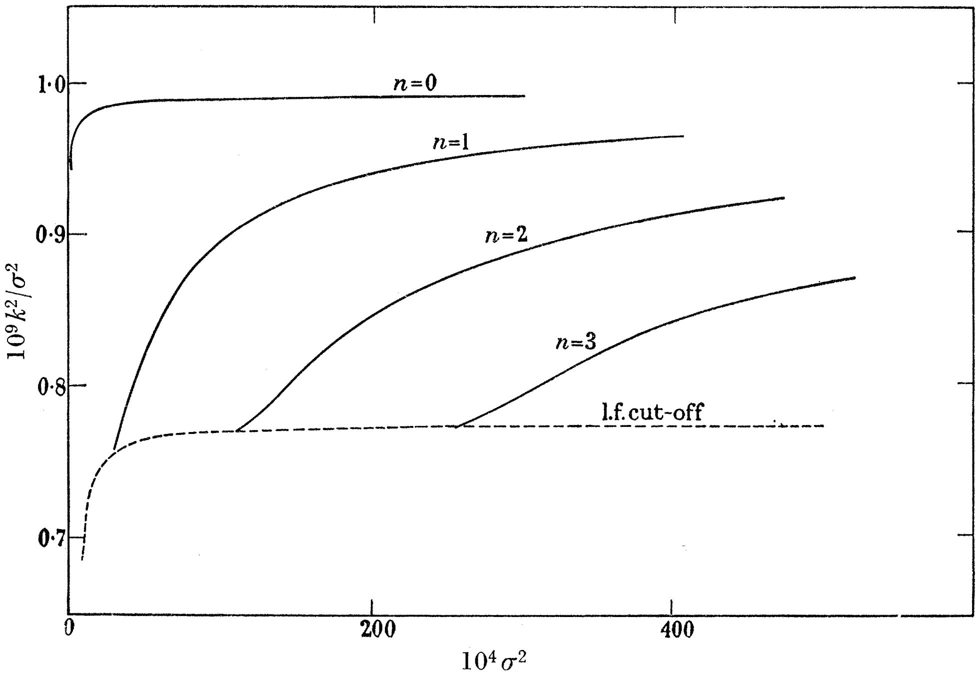

FIGURE 2. The free oscillations of a simple two-layer atmosphere consisting of a warm isothermal layer above 40 km at temperature 320 °K and a cooler isothermal layer below 40 km at temperature 250 °K.

An isothermal atmosphere can carry a 'freely travelling' pulse in which the vertical velocity is everywhere zero. The harmonic components are non-dispersive and the pulse travels like a sound pulse in two dimensions. This type of pulse, however, is 'non-physical'. The wave front is vertical and extends to infinite height and the total kinetic energy is infinite.

When $T_s > T_t$, we have a warm layer above a cold layer, and this situation represents some of the properties of the real atmosphere which are due to the warm layers which exist in the upper stratosphere (see figure 1). In this case, we find that for large σ^2

$$1/c_t^2 > (k^*/\sigma)^2 > 1/c_s^2,$$

so that k^* remains real however large the value of σ. However, for sufficiently large σ, D_t is imaginary while $D_t \coth(D_t z_2)$ remains real and many valued. It follows that there is no high-frequency cut-off, and that for large σ, there are several roots $k_n^*(\sigma)$ satisfying the boundary conditions. If we draw a graph of $(k^*/\sigma)^2$ against σ^2 for the various branches of the solution as in figure 2, it can be seen that for sufficiently large values of σ^2, the roots all lie within this range, and that all branches except the first have a low-frequency cut-off.

The physical significance of the different branches of the solution lies in the number of nodes in the velocity and pressure perturbations in the 'troposphere'. Along the first branch, the radical in (36) is real and the solution for ϖ_1 consists of two exponentials in z, and therefore has no nodes in this lower layer of the atmosphere. Along the remaining branches, this radical is imaginary and ϖ_1 is an oscillatory function of z.

For any given σ, there are only a limited number of modes of oscillation, but the number of modes increases indefinitely with σ. Figure 3 shows the variation of the modified pressure amplitude with height for the first four possible modes of oscillation when $\sigma^2 = 0{\cdot}03$, showing how in each mode the solution joins on at the 'tropopause' to a decreasing exponential solution in the 'stratosphere'. Table 3 gives a few illustrative values of the free-wave characteristics for the first four branches.

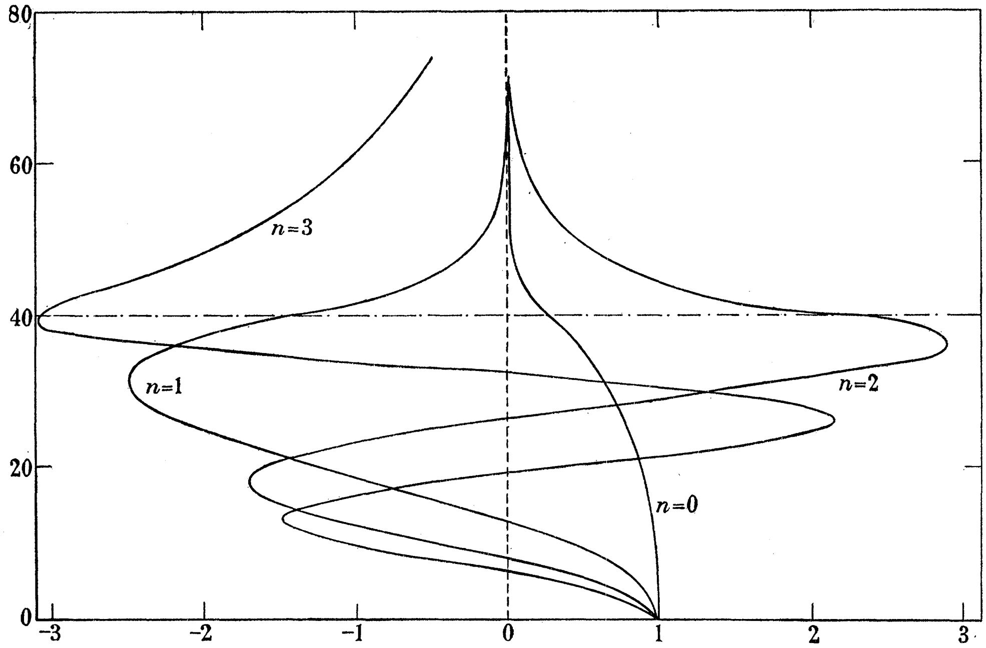

FIGURE 3. The modified pressure, on an arbitrary scale, as a function of height for the first four branches of the model atmosphere used in figure 2, for the case where $\sigma = 0{\cdot}173$, i.e. period $2\pi/\sigma = 36{\cdot}3$ s.

One can obtain an asymptotic solution of (37) for large σ for the branch with n vertical nodes, provided n is not too small.

By writing
$$k_n^2/\sigma^2 = 1/c_t^2 - \alpha_n^2/\sigma^2 \tag{39}$$
and using the fact that coth $(D_t z_2)$ has zeros at approximately $(2n+1)\,\mathrm{i}\pi/2$, it follows that
$$\alpha_n = (2n+1)\,\pi/2z_2. \tag{40}$$

Thus, for example, with $z_2 = 40$ km, and for harmonic oscillations of period 10 s, the oscillation with ten nodes on any vertical line has a k/σ value which is about 8 % less than that of the oscillation which has no node on a vertical line.

These results conflict with a conjecture made by Pekeris (1948) that all stratified atmospheres having a temperature minimum would exhibit a high-frequency cut-off. The condition that there is a high-frequency cut-off, it would seem to us, is that there is no sound channel, i.e. that at no height in the atmosphere does the temperature increase in the upwards direction. If there is a temperature minimum, then there is a sound channel, and high-frequency freely travelling waves are possible. However, in this case, oscillations with nodes on one or more horizontal planes will have low-frequency cut-offs, and even the mode

Table 3. Free-wave characteristics along branches of a type B atmosphere (c.g.s. units)

temperature −23 °C from ground level to 40 km; temperature 47 °C above 40 km

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\, dk^*/d\sigma$
	branch $n = 0$	
0	3·0729	3·0729
2	3·1056	3·1296
4	3·1156	3·1452
6	3·1219	3·1486
8	3·1258	3·1511
10	3·1286	3·1526
12	3·1308	3·1536
14	3·1326	3·1543
16	3·1340	3·1547
18	3·1352	3·1550
20	3·1362	3·1552
22	3·1371	3·1554
24	3·1379	3·1555
26	3·1386	3·1556
28	3·1392	3·1556
30	3·1398	3·1556
32	3·1403	3·1556
34	3·1408	3·1556
36	3·1412	3·1556
38	3·1416	3·1556
40	3·1419	3·1556
42	3·1423	3·1556
44	3·1426	3·1557
46	3·1429	3·1557
48	3·1431	3·1556
50	3·1434	3·1556
60	3·1441	3·1553
70	3·1452	3·1551
80	3·1458	3·1549
90	3·1463	3·1547
100	3·1468	3·1545
110	3·1471	3·1543
120	3·1474	3·1542
140	3·1480	3·1540
160	3·1484	3·1539
180	3·1487	3·1538
200	3·1489	3·1537
220	3·1492	3·1536
240	3·1494	3·1536
	branch $n = 1$	
29·39	2·7505	2·8975
30	2·7519	2·9403
32	2·7605	3·1000
34	2·7723	3·1986
36	2·7850	3·2530
38	2·7977	3·2887
40	2·8103	3·3027
42	2·8223	3·3129
44	2·8336	3·3202
46	2·8444	3·3246

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\, dk^*/d\sigma$
	branch $n = 1$ (*cont.*)	
48	2·8545	3·3259
50	2·8641	3·3271
60	2·9042	3·3194
70	2·9347	3·3074
80	2·9584	3·2952
90	2·9773	3·2685
100	2·9928	3·2732
110	3·0057	3·2653
120	3·0166	3·1578
140	3·0341	3·2461
160	3·0475	3·2360
180	3·0581	3·2279
200	3·0666	3·2190
220	3·0736	3·2146
240	3·0796	3·2105
	branch $n = 2$	
114·5	2·7775	2·8302
119·0	2·7822	3·1368
130	2·8019	3·3071
150	2·8386	3·3519
170	2·8700	3·3540
200	2·9075	3·3418
240	2·9445	3·3179
260	2·9591	3·3084
280	2·9717	3·2998
300	2·9828	3·2918
350	3·0053	3·2774
400	3·0225	3·2648
450	3·0360	3·2540
500	3·0469	3·2449
550	3·0559	3·2373
600	3·0635	3·2307
650	3·0700	3·2250
700	3·0755	3·2201
750	3·0803	3·2165
800	3·0846	3·2147
	branch $n = 3$	
256·3	2·7822	2·8219
257·0	2·7823	2·9471
260	2·7836	3·1450
270	2·7915	3·3069
280	2·8008	3·3384
300	2·8202	3·3780
600	2·9759	3·3097
650	2·9887	3·2989
700	2·9998	3·2896
750	3·0095	3·2813
800	3·0180	3·2737

$n = 0$ may have a low-frequency cut-off greater than zero (for example, compare the first entry in table 5 with the first entry in table 6).

Another feature worthy of comment in the results shown in table 3 is that $dk^*/d\sigma$ has a maximum in each branch, and for the higher branches this maximum moves towards the low-frequency cut-off. The significance of the result is that for each branch there is a σ value for which the group velocity is least. Formulae (39) and (40) do not give a minimum group velocity but are nevertheless useful in some of the discussions given later in the paper.

TABLE 4. LOW-FREQUENCY CUT-OFFS FOR THE TYPE B ATMOSPHERE OF TABLE 3 (FIRST FOUR BRANCHES ONLY)

	$10^4\sigma^2$	$10^5(k^*/\sigma)_c$	$10^5(dk^*/d\sigma)_c$
branch $n=0$	0	3·0729	3·0729
branch $n=1$	29·39	2·7505	2·8975
branch $n=2$	114·5	2·7775	2·8302
branch $n=3$	256·3	2·7822	2·8219

6. THREE-LAYER MODEL WITH SOUND CHANNEL

The two-layer model considered in the preceding section is satisfactory in the sense that the lower layer is a sound channel. However, it was thought that a three-layer model would be a better approximation to the real atmosphere. A still better approximation would be a five-layer model, with two sound channels; but, in principle, the five-layer model will contain nothing new.

The three-layer model which we choose has an isothermal troposphere at temperature T_t extending from the ground to a height z_2. The lower isothermal layer of the stratosphere has a temperature T_s, and extends from height z_2 up to height z_3. The upper layer has a temperature T_u and extends from height z_3 to infinite height. In order to represent the real atmosphere as closely as possible, the temperatures are chosen such that

$$T_u > T_t > T_s.$$

Two cases were evaluated numerically, the temperatures being the same, but the depths of the layers were varied. We take

$$T_u = 47\,^\circ\mathrm{C}, \quad T_s = -44\,^\circ\mathrm{C}, \quad T_t = -15\,^\circ\mathrm{C}, \tag{41}$$

and in the first case we take $z_2 = 10\,\mathrm{km}, \quad z_3 = 40\,\mathrm{km}$ (42)

and in the second case we take

$$z_2 = 15\,\mathrm{km}, \quad z_3 = 30\,\mathrm{km}. \tag{43}$$

In the troposphere $\varpi_1 = E(-\lambda_t e^{\kappa_t z} + \kappa_t e^{\lambda_t z}),$ (44)

where κ_t and λ_t are given by (36), the upper positive sign referring to κ_t and the lower negative sign to λ_t. The form of (44) automatically satisfies the condition of zero vertical velocity at the ground $z_1 = 0$. In the lower stratosphere

$$\varpi_1 = F e^{\kappa_s z} + G e^{\lambda_s z}, \tag{45}$$

where κ_s and λ_s are similar to κ_t and λ_t except that c_s replaces c_t.

In the upper stratosphere

$$\varpi_1 = H\,e^{\lambda_u z}, \tag{46}$$

where λ_u is similar to λ_s or λ_t except that the velocity of sound is c_u rather than c_s or c_t.

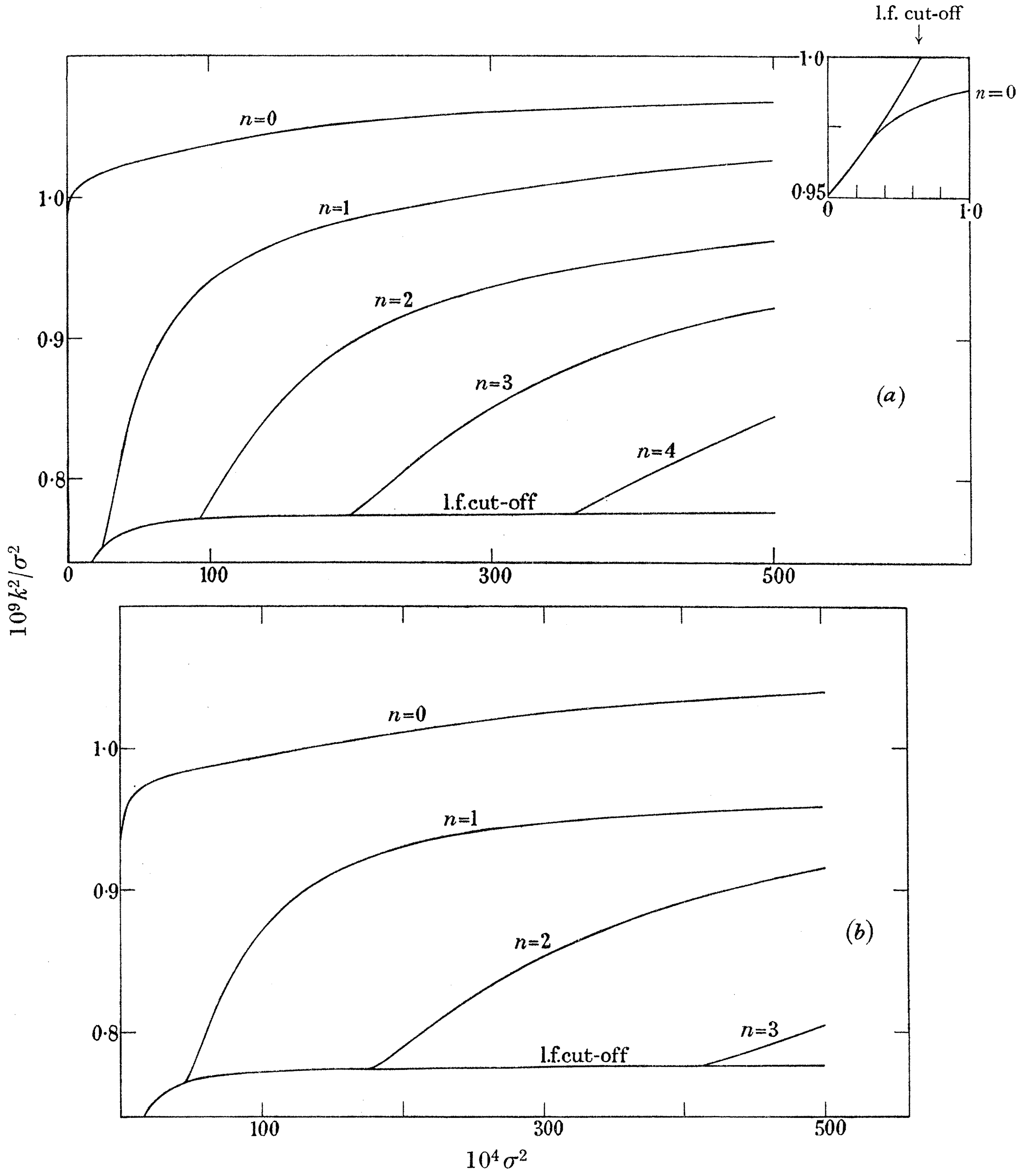

FIGURE 4. The free oscillations of three-layer atmospheres consisting of an uppermost layer at temperature 320 °K, a middle layer at temperature 229 °K and a bottom layer at temperature 258 °K. In (*a*) the interfaces are at heights 10 and 40 km and in (*b*) they are at heights 15 and 30 km.

Only the negative exponential appears in (46) because of the condition that the oscillations must have a finite total energy in any vertical column.

We require continuity at $z = z_2$ and at $z = z_3$ in both

$$\varpi_1 \quad \text{and} \quad \varpi_1'/(g\tau_0' + \sigma^2\tau_0).$$

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Eliminating the ratios E/H, F/H and G/H from the four continuity equations we obtain the equation

$$\lambda_u \lambda_t \kappa_t AB + C\lambda_s \kappa_s + \{(2-\gamma)\, g/2c_s^2\}\,(\lambda_u AC - \lambda_t \kappa_t B) - D_t(\lambda_u AC + \lambda_t \kappa_t B) \coth\{D_t(z_3 - z_2)\} = 0, \quad (47)$$

where

$$\left.\begin{aligned} A &= \{c_u^4[(\gamma-1)\, g^2 - c_s^2\sigma^2]/c_s^4[(\gamma-1)\, g^2 - c_u^2\sigma^2]\} \exp\{(1-\gamma)\, gz_3(c_s^{-2} - c_u^{-2})\},\\ B &= \{c_t^4[(\gamma-1)\, g^2 - c_s^2\sigma^2]/c_s^4[(\gamma-1)\, g^2 - c_t^2\sigma^2]\} \exp\{(1-\gamma)\, gz_2(c_s^{-2} - c_t^{-2})\},\\ C &= D_t \coth(D_t z_2) - (2-\gamma)\, g/2c_t^2. \end{aligned}\right\} \quad (48)$$

Equation (47) can be solved numerically to give k^* and $\mathrm{d}k^*/\mathrm{d}\sigma$ as functions of σ. The larger the value of σ, the more branches there are. The various branches have a different number of zeros of $\varpi_1(z)$, the first branch having no nodes (except $z = \infty$), the second branch having one node (and another at $z = \infty$), and so on.

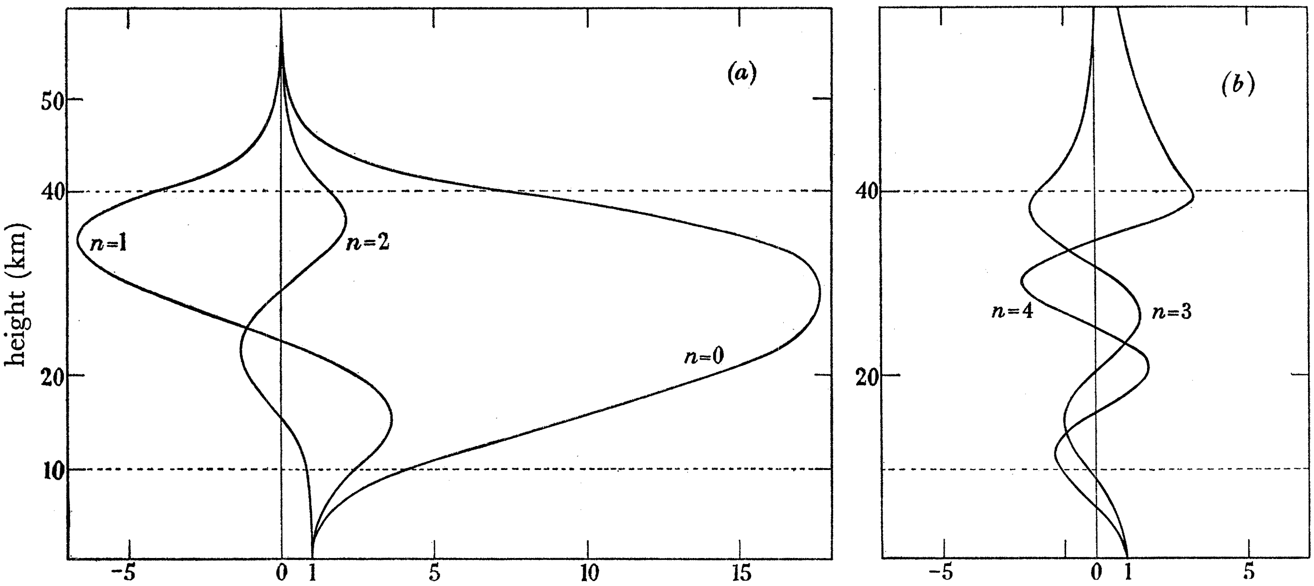

FIGURE 5. The modified pressure, on an arbitrary scale, as a function of height for the first five branches of the atmosphere used in figure 3(*a*) for the case where $\sigma = 0{\cdot}2$, i.e. period $2\pi/\sigma = 31{\cdot}4$ s. Figure 5(*a*) gives branches $n = 0$, 1, 2 and (*b*) branches $n = 3$, 4.

Figure 4 shows a plot of $(k^*/\sigma)^2$ against σ^2 for the first few branches for the two cases (42) and (43) which we studied numerically, using the I.B.M. 704 machine. Table 5 gives detailed values for a number of values of σ^2 in case (42) and table 6 gives values in case (43). The point might be noted that in case (42) all branches, including the first, have a low-frequency cut-off. The first branch has a low-frequency cut-off at $\sigma^2 = 3 \times 10^{-5}$. In case (43), all branches except the first have a low-frequency cut-off, but the first branch extends over the complete range $\sigma = 0$ to $\sigma = \infty$.

An asymptotic solution for large σ in the nth branch, the first branch being called $n = 0$, is obtained by writing

$$k_n^2/\sigma^2 = 1/c_s^2 - \alpha_n^2/\sigma^2 \quad (49)$$

and approximating to the zeros of the function $\coth\{D_s(z_3 - z_2)\}$. It is found that

$$\alpha_n = (2n+1)\, \pi/2(z_3 - z_2). \quad (50)$$

Table 5. Free-wave characteristics for the first five branches of a three-layer atmosphere corresponding to (41) and (42) (c.g.s. units)

branch $n=0$

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 dk^*/d\sigma$
0·31	3·1159	3·1609
1	3·1438	3·1803
2	3·1555	3·1864
4	3·1655	3·1924
6	3·1708	3·1945
8	3·1743	3·1964
10	3·1769	3·1980
15	3·1818	3·2015
20	3·1853	3·2049
30	3·1909	3·2117
40	3·1957	3·2188
50	3·2000	3·2259
60	3·2041	3·2330
80	3·2116	3·2462
100	3·2183	3·2574
150	3·2319	3·2764
200	3·2417	3·2864
250	3·2489	3·2917
300	3·2543	3·2946
400	3·2619	3·2973
500	3·2668	3·2982
600	3·2704	3·2985
800	3·2750	3·2984
1000	3·2780	3·2980
2000	3·2844	3·2962
4000	3·2879	3·2945
6000	3·2891	3·2938
8000	3·2897	3·2933
10000	3·2901	3·2931

branch $n=1$

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 dk^*/d\sigma$
25·80	2·7462	2·9004
26	2·7470	2·9550
28	2·7618	3·2332
30	2·7816	3·3348
35	2·8310	3·4082
40	2·8731	3·4171
45	2·9073	3·4098
50	2·9356	3·3989
60	2·9785	3·3768
80	3·0329	3·3425
100	3·0654	3·3199
150	3·1096	3·2931
200	3·1341	3·2880
250	3·1516	3·2914
300	3·1657	3·2969
400	3·1874	3·3059
500	3·2032	3·3106
600	3·2150	3·3126
800	3·2311	3·3134
1000	3·2416	3·3126
2000	3·2646	3·3073
4000	3·2774	3·3017
6000	3·2819	3·2991
8000	3·2842	3·2976
10000	3·2857	3·2966

branch $n=2$

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 dk^*/d\sigma$
92·58	2·7771	2·8846
94	2·7793	3·0260
96	2·7840	3·1171
98	2·7896	3·1660
100	2·7956	3·1969
105	2·8113	3·2399
110	2·8268	3·2624
115	2·8416	3·2751
120	2·8557	3·2832
130	2·8813	3·2915
140	2·9037	3·2950
150	2·9234	3·2961
200	2·9927	3·2911
250	3·0332	3·2847
300	3·0591	3·2752
350	3·0773	3·2834
400	3·0912	3·2894
500	3·1131	3·3074
600	3·1314	3·3237
800	3·1608	3·3368
1000	3·1821	3·3377
2000	3·2317	3·3264
4000	2·2599	3·3138
6000	3·2699	3·3080
8000	3·2751	3·3047
10000	3·2783	3·3025

branch $n=3$

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 dk^*/d\sigma$
201·24	2·7829	2·8822
202	2·7834	2·9413
204	2·7853	3·0229
206	2·7878	3·0676
208	2·7907	3·0952
210	2·7937	3·1153
215	2·8015	3·1474
220	2·8096	3·1676
225	2·8176	3·1811
230	2·8256	3·1913
240	2·8410	3·2052
250	2·8556	3·2144
260	2·8692	3·2208
280	2·8940	3·2290
300	2·9158	3·2336
350	2·9595	3·2370
400	2·9917	3·2346
500	3·0344	3·2249
600	3·0597	3·2216
800	3·0883	3·2547
1000	3·1094	3·3180
2000	3·1859	3·3535
4000	3·2353	3·3311
6000	3·2531	3·3207
8000	3·2623	3·3147
10000	3·2679	3·3108

TABLE 5 (*cont.*)

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 \mathrm{d}k^*/\mathrm{d}\sigma$	$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5 \mathrm{d}k^*/\mathrm{d}\sigma$
	branch $n=4$			branch $n=4$ (*cont.*)	
356·94	2·7850	2·9118	460	2·8753	3·2836
358	2·7854	2·9822	480	2·8902	3·2871
360	2·7866	3·0556	500	2·9039	3·2891
365	2·7905	3·1372	600	2·9594	3·2891
370	2·7949	3·1767	700	2·9990	3·2747
375	2·7996	3·2005	800	3·0276	3·2526
380	2·8044	3·2170	900	3·0483	3·2246
385	2·8093	3·2292	1000	3·0629	3·1992
390	2·8141	3·2387	2000	3·1294	3·3747
400	2·8237	3·2524	3000	3·1759	3·3680
410	2·8331	3·2617	4000	3·2038	3·3537
420	2·8421	3·2688	6000	3·2315	3·3372
440	2·8593	3·2779	8000	3·2458	3·3278
			10000	3·2546	3·3217

7. Pressure oscillations due to a source at ground level

In this section, we shall obtain a formula for the pressure oscillations at ground level caused by the introduction of a certain volume of air into the atmosphere at ground level. We shall then calculate numerical values at various distances for two-layer model atmospheres, of the type used by Scorer (1950), taking the effect of an explosion of E calories as equivalent to introducing about $8E\,\mathrm{cm}^3$ of air into the atmosphere. From these numerical values, and using published microbarograms of the Siberian meteorite and of certain nuclear explosions, we obtain approximate values for the size of the explosions. The reliability is not good for several reasons. First, it is not clear which model atmosphere approximates best to the atmosphere existing at the time of the explosion, or what corrections must be made for winds. Second, the model atmospheres have a cut-off frequency which is low, and therefore only comparatively long waves are emitted. Third, and this is perhaps the most important point, the theory predicts that the amplitudes of the waves are linearly proportional to the energy of the explosion, and the periods are independent of the size of the explosion. The theory disagrees with observations on both the second and third points.

Pekeris and Scorer have constructed Fourier integrals, using the fundamental solutions corresponding with the oscillations of the two-layer model atmosphere with a cold stratosphere, to represent the freely travelling pulse caused by a source which has a singularity in the vertical velocity at ground level at time zero. We found it difficult to understand the physical significance of their integrals and we hope that the following discussion will help to clarify the situation.

The system of co-ordinates which we have adopted, and which are convenient for treating the freely travelling waves, are not those which would naturally be used for calculating the sound pulse sent out by a source on a plane which is the boundary of an isotropic atmosphere filling all space on one side of the plane. If this plane is regarded as 'horizontal' one would naturally use spherical polar co-ordinates with the origin at the source in the plane; the polar axis being 'vertical'. A source which was introducing air by a singularity in the vertical velocity at the origin would also be introducing air by a horizontal singularity at the origin, and the source would be uniform though a hemisphere of small radius.

Table 6. Free-wave characteristics for the first four branches of a three-layer atmosphere corresponding to (41) and (43) (c.g.s. units)

$10^4\sigma^2$	$10^5 k^*/\sigma$	$10^5\,\mathrm{d}k^*/\mathrm{d}\sigma$
	branch $n=0$	
0	3·0540	3·0540
2	3·0814	3·1144
4	3·0938	3·1313
6	3·1014	3·1390
8	3·1067	3·1435
10	3·1108	3·1466
15	3·1178	3·1509
20	3·1226	3·1536
30	3·1291	3·1570
40	3·1337	3·1597
50	3·1374	3·1621
60	3·1408	3·1646
80	3·1468	3·1700
100	3·1525	3·1759
150	3·1663	3·1929
200	3·1791	3·2103
250	3·1903	3·2257
300	3·1998	3·2383
400	3·2145	3·2564
500	3·2251	3·2679
600	3·2330	3·2756
800	3·2440	3·2847
1000	3·2514	3·2896
2000	3·2685	3·2965
3000	3·2752	3·2973
4000	3·2788	3·2971
6000	3·2827	3·2964
8000	3·2848	3·2957
10000	3·2860	3·2952
	branch $n=1$	
44·80	2·7651	2·8201
46	2·7674	2·8316
48	2·7738	2·8423
50	2·7818	2·8493
55	2·8043	2·8636
60	2·8270	2·8781
65	2·8482	2·8931
70	2·8678	2·9081
80	2·9011	2·9367
90	2·9284	2·9621
100	2·9507	2·9842
120	2·9847	3·0196
140	3·0088	3·0457
160	3·0264	3·0651
180	3·0396	3·0798
200	3·0498	3·0911
250	3·0669	3·1106
300	3·0773	3·1236
400	3·0896	3·1441
500	3·0976	3·1665
600	3·1050	3·1965
800	3·1231	3·2609
1000	3·1432	3·2886
2000	3·2023	3·3120
3000	3·2275	3·3138
	branch $n=1$ (*cont.*)	
4000	3·2414	3·3128
6000	3·2564	3·3098
8000	3·2644	3·3073
10000	3·2694	3·3054
	branch $n=2$	
177·40	2·7822	2·7994
178	2·7825	2·8024
180	2·7837	2·8112
185	2·7884	2·8289
190	2·7943	2·8439
195	2·8008	2·8574
200	2·8075	2·8698
210	2·8211	2·8924
220	2·8344	2·9124
230	2·8472	2·9303
240	2·8593	2·9466
260	2·8817	2·9750
280	2·9016	2·9988
300	2·9194	3·0191
350	2·9561	3·0581
400	2·9846	3·0851
450	3·0073	3·1032
500	3·0255	3·1142
600	3·0523	3·1652
800	3·0798	3·2104
1000	3·0896	3·2373
2000	3·1207	3·3117
3000	3·1507	3·3413
4000	3·1801	3·3404
6000	3·2128	3·3333
8000	3·2306	3·3273
10000	3·2417	3·3228
	branch $n=3$	
414·92	2·7853	2·8165
416	2·7856	2·8277
418	2·7863	2·8416
420	2·7871	2·8511
425	2·7896	2·8659
430	2·7924	2·8755
435	2·7955	2·8826
440	2·7986	2·8885
450	2·8050	2·8981
460	2·8115	2·9063
480	2·8243	2·9207
500	2·8365	2·9336
600	2·8726	2·9801
800	2·9161	3·0436
1000	2·9462	3·0885
2000	3·0295	3·2392
3000	3·0723	3·3110
4000	3·1048	3·3488
6000	3·1531	3·3647
8000	3·1836	3·3558
10000	3·2031	3·3477

Before we attempt to construct Fourier integrals to represent the disturbance caused by a point source at ground level in model atmospheres (gravity present), it will be illuminating to consider the simpler problem of an isotropic semi-infinite atmosphere with no gravity present. The object here is to ensure that the spherical waves which are known to be the solution are in fact properly represented by Fourier integrals in the system of co-ordinates which we have chosen.

When there is no gravity, we have as one choice of fundamental solutions

$$\left.\begin{aligned} \varpi &= \tau_0 \sigma \sin \sigma t\, J_0(kr)\, (\sin \beta z)/\beta, \\ w &= \cos \sigma t\, J_0(kr) \cos \beta z, \\ u &= k \cos \sigma t\, J_0'(kr)\, (\sin \beta z)/\beta, \\ \beta^2 &= \sigma^2/c^2 - k^2. \end{aligned}\right\} \tag{51}$$

When $k > \sigma/c$, $\cos \beta z$ is replaced by $e^{-\beta_1 z}$ and $\sin \beta z$ by $-e^{-\beta_1 z}$, where $\beta_1^2 = k^2 - \sigma^2/c^2$ and $\beta_1 > 0$. The parameter τ_0 is the reduced temperature given by (1), i.e. $\tau_0 = 1/T_0$, where T_0 is the temperature of the isotropic atmosphere being considered.

Fourier integrals representing a singularity at the origin are

$$\left.\begin{aligned} \varpi &= B\tau_0 \int_0^\infty d\sigma \int_0^\infty dk \Sigma \sigma k \sin \sigma t J_0(kr)\, (\sin \beta z)/\beta, \\ u &= B \int_0^\infty d\sigma \int_0^\infty dk \Sigma k^2 \cos \sigma t J_0'(kr)\, (\sin \beta z)/\beta, \\ w &= B \int_0^\infty d\sigma \int_0^\infty dk \Sigma k \cos \sigma t J_0(kr) \cos \beta z, \end{aligned}\right\} \tag{52}$$

where B is a constant which will be determined by the volume which the source introduces into the atmosphere. The function Σ is an arbitrary function of σ which, as we shall show, determines the time variation of the source.

Consider the expression for ϖ. Replace $J_0(kr)$ by its asymptotic form for large kr, express all trigonometrical functions in their exponential forms, take the products and convert back again to get travelling waves. Then

$$\varpi = -B\{\tau_0/2(2\pi r)^{\frac{1}{2}}\} \int_0^\infty d\sigma \int_0^\infty dk \sigma \Sigma (k^{\frac{1}{2}}/\beta) \sin (\sigma t - kr - \beta z - \tfrac{1}{4}\pi). \tag{53}$$

In this expression, we have omitted all of the terms except that which has the combination $(\sigma t - kr - \beta z - \frac{1}{4}\pi)$ for the reason that the argument now to be developed shows that these other terms make zero contribution to the integral.

From the principle of stationary phase, the rapid oscillations of the integral give cancellation except from a small region of the positive (σ, k) quadrant such that $(\sigma t - kr - \beta z)$ is stationary. Consequently, ϖ at (r, z, t) comes from the neighbourhood of σ and k values such that

$$\left.\begin{aligned} t &= z\partial\beta/\partial\sigma = z\sigma/c^2\beta, \\ r &= -z\partial\beta/\partial k = zk/\beta. \end{aligned}\right\} \tag{54}$$

The exponential terms neglected in (53) never have a stationary phase in the positive (σ, k) quadrant, and therefore cancel to zero.

From (54), it follows that

$$r/z = k/\beta, \quad k = \kappa \sin\theta, \quad \beta = \kappa\cos\theta,$$
$$\kappa^2 = \beta^2 + k^2 = \sigma^2/c^2.$$

Writing $\qquad r = R\sin\theta, \quad z = R\cos\theta, \quad R^2 = r^2+z^2,$

we have $\qquad R = ct.$

Expanding the phase in the neighbourhood of its stationary value, we have

$$\sigma t - kr - \beta z = \sigma(t-R/c) - \tfrac{1}{2}z\{(\partial^2\beta/\partial\sigma^2)\,\delta\sigma^2 + \ldots\} + \ldots$$
$$= \sigma(t-R/c) + (zk^4/2c^2\beta^3)\,\{\delta(\sigma/k)\}^2 + \ldots.$$

We now change from the independent variables (σ, k) to the independent variables $(\sigma, \sigma/k)$, the Jacobian being $-\sigma/k^2$. Then

$$\varpi = -B\{\tau_0/2(2\pi r)^{\frac{1}{2}}\}\int_0^\infty \mathrm{d}\sigma \int_{-\infty}^{\infty} (k^{\frac{5}{2}}/\beta)\,\Sigma \sin\{\sigma(t-R/c) - \tfrac{1}{4}\pi + \alpha^2\phi^2\}\,\mathrm{d}\phi,$$

where $\qquad \alpha^2 = zk^4/2c^2\beta^3.$

Splitting the last factor into two components, and using standard formulae for the ϕ integration, we obtain

$$\varpi = -(B\tau_0/2R)\int_0^\infty \sigma\Sigma \sin\sigma(t-R/c)\,\mathrm{d}\sigma. \tag{55}$$

In the theory of sound in an infinite isotropic medium, suppose that there is a point source which is introducing volume at a rate $2Vf(t)$, where $\int_0^\infty f(t)\,\mathrm{d}t = 1$. Then the spherical sound wave being emitted has a pressure disturbance

$$\delta p = (\rho_0 V/2\pi R)\,f'(t-R/c).$$

Compare this with the pressure disturbance given by (55),

$$\delta p = -(B\rho_0/2R)\int_0^\infty \sigma\Sigma\sin\sigma(t-R/c)\,\mathrm{d}\sigma.$$

The two expressions are the same provided that Σ is normalized such that

$$\int_0^\infty \mathrm{d}t \int_0^\infty \Sigma\cos\sigma t\,\mathrm{d}\sigma = 1 \tag{56}$$

in which case $\qquad B = V/\pi.$

The source which is introducing volume at the rate $2Vf(t)$ into the full space gives the same pressure disturbance as a source which is introducing a volume $Vf(t)$ at a point on a plane with the medium only on one side, i.e. into the half space.

Thus, the expressions (52) do correctly give the hemispherical sound wave caused by a point source on the boundary plane of a semi-infinite isotropic atmosphere; and if the total volume introduced by the source is V the value of B is V/π. The function Σ must be normalized according to (56).

Suppose that at a certain distance Z from the 'ground' plane where the source operates, there is a change from one isotropic medium to another—for example, a discontinuous

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change of temperature. The pulse sent out by the source is reflected and refracted at the interface; and in due time there will be secondary and higher reflexions and refractions. If we constructed Fourier integrals to represent the pulse, we would extend (52) in the region 1 by adding to $\cos\beta z$ in the expression for w, terms such as

$$A\{\cos\beta(2Z-z)-\cos\beta(2Z+z)\}.$$

The continuity conditions on w and ϖ at the interface would determine the coefficients $A, \ldots$. The solution is no doubt the same as that obtained by using the theory of sound and the method of images; but the amplitudes variations along the wave fronts are not easily calculated by either method.

We shall now consider model atmospheres with gravity present.

We have as fundamental solutions

$$\left.\begin{aligned} w &= \cos\sigma t J_0(kr)\,\partial\varpi_1(z)/\partial z,\\ \varpi &= \{(\sigma^2\tau_0+g\tau_0')/\sigma\}\sin\sigma t J_0(kr)\,\varpi_1(z).\end{aligned}\right\} \tag{57}$$

The functions $\varpi_1(z)$ conform to the equations of motion and of continuity, and to the continuity conditions at the interfaces between layers; but the condition that the vertical velocity at the ground is zero is not necessarily obeyed. The functions $\varpi_1(z)$ are defined over the whole of the positive (σ, k) quadrant. They include as special cases the solutions representing the modes which are confined to the lower layers and which are the modes which give the freely travelling gravity waves.

Generalizing these solutions, we take

$$w = (V/\pi)\int_0^\infty \mathrm{d}\sigma\int_0^\infty \mathrm{d}k\Sigma k\cos\sigma t J_0(kr)\,\varpi_1'(z)/\varpi_1'(z_1),$$

$$\varpi = (V/\pi)\int_0^\infty \mathrm{d}\sigma\int_0^\infty \mathrm{d}k\{(\sigma^2\tau_0+g\tau_0')/\sigma\}\,\Sigma k\sin\sigma t J_0(kr)\,\varpi_1(z)/\varpi_1'(z_1), \tag{58}$$

where z_1 is the value of z at ground level and V is the volume introduced by the singularity.

The function Σ is normalized according to (56). Analytical forms of Σ convenient for our purpose are

$$\Sigma = (2/\pi)\,\mathrm{e}^{-\sigma T}, \quad f(t) = 2T/\pi(T^2+t^2), \tag{59a}$$

$$\Sigma = (2/\pi)\,(1-\sigma T)\,\mathrm{e}^{-\sigma T}, \quad f(t) = 4Tt^2/\pi(T^2+t^2)^2, \tag{59b}$$

It might be helpful here to make a brief digression on the source function Σ to represent an explosive source. The case of (59*b*) is a reasonable approximation; a better approximation is to take the pressure–time curve from an explosion in 'free air', and actually calculate numerically the function Σ which represents it.

Suppose that for an explosion in 'free air' the pressure disturbance p, at radius r and time t, measured from the arrival of the blast, is $p(r, t)$. The radius r must be taken sufficiently large for the blast wave to be of small intensity, in order that the theory of sound will apply to the subsequent motion. Then the volume V introduced into the air, to give the same pulse at distances greater than r is

$$V = (4\pi r/\rho_0)\int_0^{\tau_1}\mathrm{d}t\int_0^t p(r, t)\,\mathrm{d}t,$$

where τ_1 is the time duration of the blast wave at r.

Taking some observed blast waves, and making the double integration, we find that the volume V is 5 to 8 cm³ per calorie of explosive energy, as described in the Introduction.

Figure 6 shows the functions $f(t)$ and Σ as given by (59 b); and for comparison the source functions calculated from the experimental results on the blast from explosions. The time T has been taken as equal to one-quarter of the blast duration τ_1.

The problem now is to perform the integrations over the positive (σ, k) quadrant in the formula for ϖ.

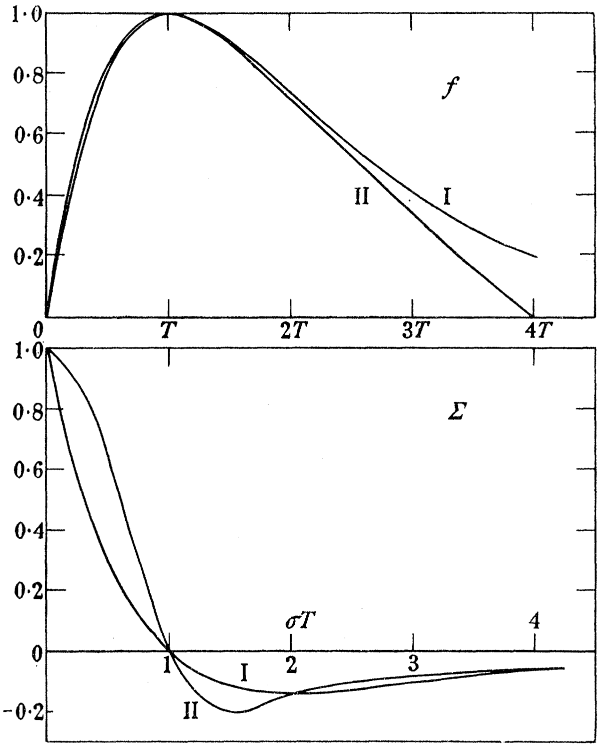


FIGURE 6. The source function $f(t)$ and its transform Σ as given by (59 b) compared with similar functions computed from blast waves caused by explosions. I refers to (59 b) and II to blast waves. The time T has been taken as one-quarter of the blast duration τ_1. The units of f are $1/\pi$, and of Σ are $2/\pi$, in order to normalize $\int_0^\infty f(t)\,dt$ to unity.

The functions $\varpi_1(z)$ will be oscillatory in some regions of the quadrant. By the same procedure as was used for the isotropic atmosphere, the contribution to the pulse will come from a synthesis of waves of the type $e^{\nu z}\, e^{i(\sigma t - kr - \beta z)}$, and the method of stationary phase shows that the pulse at (r, z, t) will come entirely from σ and k values in the neighbourhood of σ and k values satisfying

$$t = z\,\partial\beta/\partial\sigma, \quad r = -z\,\partial\beta/\partial k. \tag{60}$$

Even in the case of model atmospheres with a high-frequency cut-off, there is still a pressure disturbance at ground level due to (σ, k) values which represent the 'direct beam', travelling horizontally at ground level. The 'direct beam' is the integral of waves whose z factor is the limiting case of $e^{\nu z}(\sin \beta z)/\beta$ when β tends to zero. There are also 'direct beams' from the images of the source in the interfaces, and their images. We have not attempted to calculate these parts of the pulse; but as far as comparison with actual measurements of the waves caused by an explosion is concerned, we may ignore them because wind gradients and temperature gradients near the ground will scatter the beams into 'noise'. The freely travelling waves are much less affected by local perturbations because the whole depth of the atmosphere is partaking in the motion.

In the case of the freely travelling waves, the function $\varpi_1'(z_1)$ is zero, because the vertical velocity is zero at ground level z_1. There is a curve in the (σ, k) positive quadrant, $k = k^*(\sigma)$, or a series of such curves, along which the integrand for ϖ has a singularity.

We have for ϖ at ground level, denoted by ϖ_1, the real part of the expression

$$\varpi_1 = \{V/\pi(2\pi r)^{\frac{1}{2}}\} \int_0^\infty d\sigma \int_0^\infty dk\{(\sigma^2\tau_{01}+g\tau_{01}')/\sigma\}\Sigma k^{\frac{1}{2}}\{\varpi_1(z_1)/\varpi_1'(z_1)\}\, e^{i(\sigma t-kr-\frac{1}{4}\pi)}.$$

Waves which have no stationary phase have been omitted.

The k-integration can be performed by the method of residues. Define a contour in the complex k plane consisting of the positive real axis, the negative imaginary axis, and a curve Γ joining the two ends of the axes at infinity. The only singularity is on the real axis at $k = k^*$. For large $r|k|$, the integral along Γ is zero, owing to the rapid oscillations and the negative exponential. Along the imaginary axis, there is a contribution near the origin but this tends to zero, provided $|k|$ is small but $r|k|$ is large. Thus, asymptotically for large r we have that the contour integral, starting from the origin along the real axis, is $-\pi i$ times the residue at k^*.

Expressing ϖ_1 in terms of the pressure disturbance p_1 at ground level, we have

$$p_1 = \rho_{01}\varpi_1/\tau_{01},$$

$$p_1 = \{\rho_{01}V/(2\pi r)^{\frac{1}{2}}\} \int_0^{\sigma_e} d\sigma \Sigma F(\sigma)\,(k^*)^{\frac{1}{2}}\, e^{i(\sigma t-kr+\frac{1}{4}\pi)}, \tag{61}$$

where
$$F(\sigma) = -\{(\sigma^2\tau_{01}+g\tau_{01}')/\sigma\tau_{01}\}/\{\partial[\varpi_1'(z_1)/\varpi_1(z_1)]/\partial k\}_{k=k^*}. \tag{62}$$

Values of $F(\sigma)$ for the three atmospheres defined in table 1 are given in table 7.

For the model atmospheres where the uppermost layer is warmer than the lower layers, the pressure disturbance p_1 is the sum over all branches, and the functions $F_n(\sigma)$ must be evaluated for each branch. The range of integration for any branch is from the low-frequency cut-off to infinity.

The special case where the excitation function Σ is a constant is obtained from (61) by making T very small, and $\Sigma = 2/\pi$. In this case

$$p_1 = \{\rho_{01}V/\pi(\tfrac{1}{2}\pi r)^{\frac{1}{2}}\} \int_0^{\sigma_c} d\sigma F k^{\frac{1}{2}}\, e^{i(\sigma t-kr+\frac{1}{4}\pi)}, \tag{63}$$

where the real part is required, and for simplicity, the asterisk on k has been dropped, it being understood that k is a function of σ.

At sufficiently large values of r, the exponential in the integral oscillates rapidly and the integration can be made by the method of stationary phase. First we define a value of σ, denoted by σ_0, for which precisely

$$t = r(dk/d\sigma)_0. \tag{64}$$

Then for these precise values of r and t, we take neighbouring values $\sigma = \sigma_0+\delta\sigma$, in which case

$$\sigma t-kr+\tfrac{1}{4}\pi = r(\sigma_0 k_0'-k_0)+\tfrac{1}{4}\pi-\tfrac{1}{2}rk_0''(\delta\sigma)^2-\tfrac{1}{6}rk_0'''(\delta\sigma)^3+\dots. \tag{65}$$

We now replace σ by σ_0 in the slowly varying functions of the integral, and take these functions outside the integral sign. Moreover, we neglect the $(\delta\sigma)^3$, $(\delta\sigma)^4$, ... terms. Since r is going to be taken as large, we make the modification which allows for the curvature of the earth, as explained in the discussion of (19). Thus for $k_0'' > 0$

$$p_1 = (2\rho_{01}V/\pi)\,(1/ra\sin\theta)^{\frac{1}{2}}\,F(\sigma_0)\,(k_0/k_0'')^{\frac{1}{2}}\cos\{r(\sigma_0 k_0' - k_0)\}. \tag{66}$$

The validity of neglecting the $(\delta\sigma)^3$ term requires that $rk'''/(rk'')^{\frac{3}{2}}$ should be small compared with unity. Similarly, the neglecting of $(\delta\sigma)^4$ term requires that $rk^{\mathrm{iv}}/(rk'')^2$ should be small

Table 7. Relative intensity functions (sec^{-1}) for type A atmospheres I, II, III of table 1

	$10^3F(\sigma)$		
$10^4\sigma^2$	I	II	III
0	11·92	12·94	14·25
1	11·67	12·54	13·67
2	11·41	12·36	13·14
3	11·16	11·93	12·43
4	10·90	11·49	11·86
5	10·64	11·03	11·29
6	10·36	10·55	10·65
7	10·09	10·05	10·00
8	9·80	9·52	9·26
9	9·51	9·01	8·45
10	9·20	8·43	7·54
11	8·92	7·78	6·54
12	8·61	7·07	5·49
13	8·32	6·37	4·45
14	7·97	5·60	3·15
15	7·65	4·69	1·95
16	7·28	3·77	—
17	6·94	2·78	—
18	6·55	1·62	—
19	6·19	0·31	—
20	5·79	—	—
21	5·43	—	—
22	4·99	—	—
23	4·59	—	—
24	4·10	—	—
25	3·63	—	—
26	3·12	—	—
27	2·73	—	—
28	2·17	—	—
29	1·13	—	—
30	—	—	—

compared with unity. These conditions are only satisfied at very large distances r. For example, with atmosphere I of table 1 at $\sigma = 0{\cdot}03$, and at a distance 1000 km, $rk'''/(rk'')^{\frac{3}{2}}$ is about unity and $rk^{\mathrm{iv}}/(rk'')^2$ is also about unity.

When the method of stationary phase is approximately valid, the duration of the pressure–time record observed at range r is given by

$$t_c - t_0 = r\{(\mathrm{d}k/\mathrm{d}\sigma)_{\sigma=\sigma_c} - (\mathrm{d}k/\mathrm{d}\sigma)_{\sigma=0}\}, \tag{67}$$

where $\sigma = \sigma_c$ is the cut-off frequency for the assumed atmosphere. For the atmospheres considered in table 1, the distance-durations are given in table 8.

At distances r which are too small for the stationary-phase method to be valid, no simple analytical extension is satisfactory. It is not good enough to retain $(\delta\sigma)^3$ and drop $(\delta\sigma)^4$; and in any case, one must include at least the first gradients of $F(\sigma)$ and $k^{\frac{1}{2}}$. A direct numerical integration is the best procedure. We need numerical values of the real part of the integral

$$I = \int_0^{\sigma_c} F(\sigma)\, k^{\frac{1}{2}}\, e^{i(\sigma t - kr + \frac{1}{4}\pi)}\, d\sigma.$$

Having chosen a value for r, we then choose values for t and σ_0, guided by (64). Thus the chosen values of r, t and σ_0 satisfy (60) exactly. Then we rewrite I in the form

$$I = e^{i\{r(\sigma_0 k_0' - k_0) + \frac{1}{4}\pi\}} \int_0^{\sigma_c} F(\sigma)\, k^{\frac{1}{2}}\, e^{ir\phi}\, d\sigma,$$

where $$\phi = (k_0 - k) + (\sigma - \sigma_0)\, k_0'.$$

Table 8

atmosphere	$(t_c - t_0)/r$ (s km^{-1})
I	0·155
II	0·419
III	0·647

We now make numerical integrations to get the real and imaginary parts of the integral appearing in I. Suppose that these parts are M and N respectively. Then

$$p_1(r,t) = (\rho_{01} V/\pi)\,(2/\pi r)^{\frac{1}{2}}\,[M \cos\{r(\sigma_0 k_0' - k_0) + \tfrac{1}{4}\pi\} - N \sin\{r(\sigma_0 k_0' - k_0) + \tfrac{1}{4}\pi\}]. \qquad (68)$$

Having done the integration for this value of t, we next take a neighbouring value of t, and make a similar integration. Most of the numerical values can be used again. Proceeding in this way, we can construct the pressure–time curve at distance r.

Figures 7 and 8 show pressure–time curves at ground level caused by the introduction of 1 km^3 of air at ground level, calculated by the method just described, for atmosphere I at distances 1000 and 3500 km and for atmosphere II at distance 3500 km. For atmosphere I at distance 6400 km, the pressure–time curve was calculated by using (66). It will be seen from these figures that the pressure oscillations start earlier and finish later than the method of stationary phase would lead us to expect. However, these wings are unimportant except at the closest distance, $r = 1000$ km.

Numerical values

We have already explained that (66) and its modifications are unlikely to give a good representation of the experimental observations. Nevertheless, it is interesting to apply the formula to certain observations, and see how large are the values obtained for the energies of the explosions.

Microbarograms of the Siberian meteorite of 1908 have been published by Astapowitsch (1934) and Whipple (1930, 1934). Scorer (1950) has reproduced some of them. Yamamoto (1957) has given a useful reproduction of a number of microbarograms of nuclear explosions obtained in Japan by Shida and others.

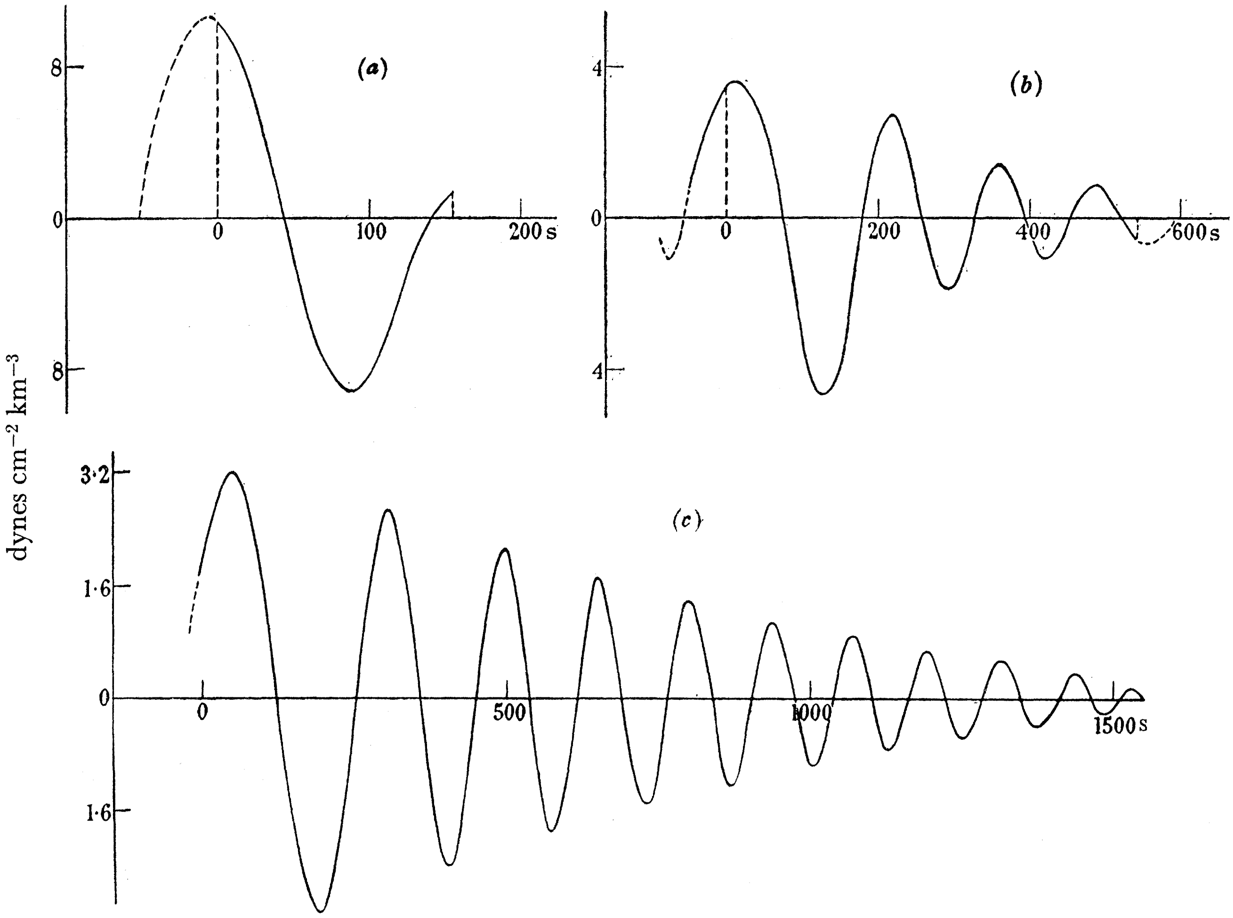


FIGURE 7. The pressure pulse on the ground in dynes cm^{-2} km^{-3} for Scorer's atmosphere (atmosphere I of table 1) at distance (*a*) 1000 km, (*b*) 3500 km and (*c*) 6400 km. The vertical dashed lines show where the records would begin and end according to the method of stationary phase.

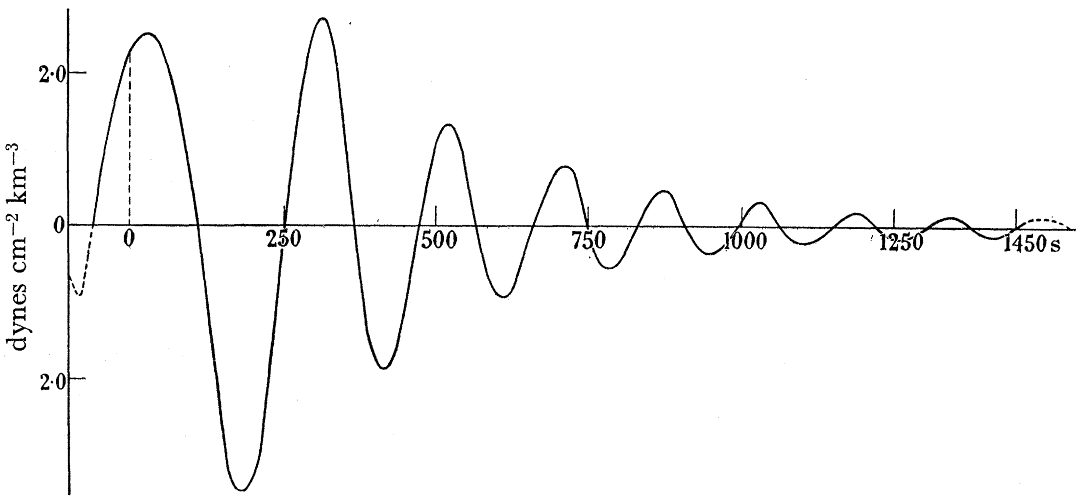


FIGURE 8. The pressure pulse at the ground at a distance 3500 km in atmosphere II of table 1. The pulse may be compared with that shown in figure 7 (*b*).

If variations in the amplitudes of the waves caused by variations in meteorological conditions over the same geographical route are small, the largest nuclear explosions appear to be those of 1 March 1954 and 5 May 1954. Each gave a peak to trough maximum amplitude of about 400 dyn/cm^2 at a distance of 3500 km.

The Siberian meteorite caused air waves at Leningrad (3740 km) with a maximum peak to trough amplitude of 360 dyn/cm^2 and at Kew (5750 km), a peak to trough amplitude of 197 dyn/cm^2.

If one compares the theoretical curve shown in figure 7, which applies to atmosphere I of table 1, with the microbarograms mentioned above, the theoretical waves last only for about 10 min, whereas the observed waves lasted for at least $\frac{1}{2}$ h. If a model atmosphere with a cold stratosphere is a suitable model, then atmosphere I should have been a good choice for calculating the waves from the meteorite. If we use figure 7, and disregard the fact that the computed duration is too small by a factor of about 3 or more, the observed amplitudes would indicate that the explosion was equivalent to the introduction of 45 km^3 of air into the atmosphere.

Taking 1 cal of explosive energy as giving an expansion of 8 cm^3 in the atmosphere, the meteorite had an energy 6×10^{15} calories or 6 MT.

Similarly, atmosphere II and figure 8 should be a good choice in the case of the air waves recorded in Japan from nuclear explosions in the Pacific. The waves are now more dispersive and the theoretical duration at 3500 km is nearly $\frac{1}{2}$ h. If we take the maximum amplitude per km^3 from figure 8, then the explosions of 1 March and 5 May 1954 were 8 to 10 MT.

Our conclusion is that a two-layer model atmosphere, which is chosen to represent average conditions up to say 25 or 30 km and having the upper layer colder than the lower layer, seems in the case of large explosions to give about the correct amplitude for the train of waves which have periods of about 1 min. However, the theoretical duration of these slow oscillations is independent of the size of the explosion; and, in contrast with experiment, these slow oscillations are not just a part of an extremely long wave train, containing waves of much shorter period. In the case of smaller explosions (10^{13} cal), the calculated waves are unlike those observed.

A digression on the meteorite

The Siberian meteorite poses a number of fascinating problems. The total energy can be roughly fixed by the observations on the destruction of trees, but there is the curious absence of an enormous crater. Whipple (1934) quotes the following description: 'the coniferous forest (taiga) was uprooted and burnt to a radius of 10 to 15 km by the action of the hot explosive wave; trees were felled by the air over a radius of 20 km and further off, chiefly on high ground, were uprooted for a distance of 40 to 50 km'.

A long-duration blast wave, such as was produced by the impact of the meteorite, would uproot coniferous trees where the overpressure was about 2 Lb./in.^{2} or more. Taking the 20 km radius, quoted by Whipple, as representing the 2 Lb./in.^{2} level, one can use the blast data for ground-burst nuclear devices (U.S.A.E.C. 1957), to estimate the size of the explosion. The value obtained is 13 MT.

The observation quoted by Whipple that trees were uprooted up to distances of 40 to 50 km, but by implication, only in a 'patchy' pattern, is probably explained by refraction and focusing of the blast, due to local thermal and wind gradients in the atmosphere.

An explosion of 13 MT on the ground would have produced a crater 300 ft. deep and half a mile in diameter (U.S.A.E.C. 1957). This is enormously greater than anything reported. Such a crater, if there, could hardly have escaped notice by the explorer Kulik (1927). Scores of craters were found, the largest perhaps originally one hundred feet across and a few tens of feet deep.

Whipple (1934) gives astronomical arguments which suggest that the meteorite was moving at 72 km/s when it entered the atmosphere. If the speed was as high as this, the meteorite probably broke into pieces under the stresses caused by the air resistance. The thrust per unit area on the 'nose' of the meteorite would be approximately ρV^2, and with $V = 70$ km/s, the thrust at the nose would be more than 400 tons/in.^{2} at sea level, and 40 tons/in.^{2} at a height of 18 km above the ground, where the air density is one-tenth of the sea level value.

At some height, a hole would have been punched through the meteorite, and the ring-shaped residue would then have broken up. Each piece would have gone through the same breaking-up process, the time-scale decreasing linearly with the size of the fragment. Provided the aerodynamic forces were much bigger than the mechanical strength of the meteorite material, the main balance would be between the aerodynamic and the inertial forces. The time taken to punch a hole through the meteorite can be obtained from a similarity principle and inserting the observed time for this to happen with liquid drops in an air current.

Mr W. R. Lane has kindly given us the following experimental data. Water drops of radii 2·0, 1·0 and 0·5 mm in air currents 12·5, 17·5 and 24·7 m/s distorted and developed into a torus in times 8·5, 2·7, 1·0 ms. If U is the air speed, T is the time for a hole to be punched through, and a is the original radius, the non-dimensional number applying to all water drops (big enough for surface tension and viscosity to be dominated by the aerodynamical forces) is UT/a, and Mr Lane's values give for this number 53, 47 and 49 respectively, an average of 50.

If the Siberian meteorite had been 15 m in radius and density 1, then at sea level in an air stream of 70 km/s, a hole would be punched through in a time 0·01 s. If the density of the meteorite had been 5 and the air density one-tenth of the sea level value, the time would be larger by a factor $(50)^{\frac{1}{2}}$, i.e. a hole would be punched through the middle in 0·07 s.

It may be of interest to note that the *Encyclopaedia Britannica* states in the article on Meteorites that a ring-shaped iron meteorite weighing 1500 lb was found at Tucson, Arizona, and that jaw-shaped pieces of iron meteorites, obviously parts of a ring, have been found in South Africa. The middles may have been pushed out by air resistance or by ground resistance. Some tektites have the shape of a ring or torus, and Mr W. R. Lane informs us that small spheres of Wood's metal can be pushed into the shape of a torus by subjecting them to air blast. Interesting photographs of water drops breaking up in air currents have been published by Green & Lane (1957).

The Siberian meteorite may well have broken up a few seconds before impact, the fragments striking the ground over a considerable area. Such a hypothesis would account for the absence of a major crater.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

8. The experimental results—mainly as reported at Geneva

A summary will now be given of the experimental results, mainly as reported at Geneva.

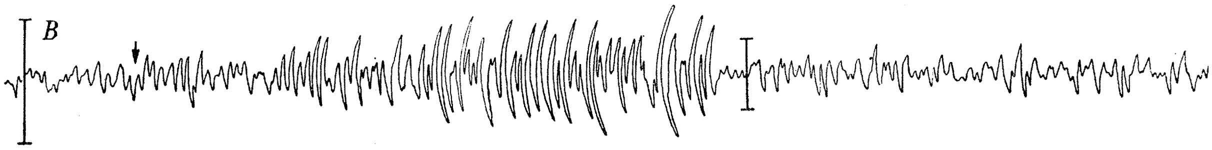
Explosions of energy 10^{23} ergs or more can be recorded at distances of a few thousand kilometres by standard microbarographs of the type often used in observatories. However, instruments of this type have a frequency response which is only adequate to record waves of periods in the range 1 to 5 min. The published records of the Siberian meteorite, and of large nuclear explosions, usually show a series of waves, lasting up to $\frac{1}{2}$ h. The amplitudes are of the order 100 dyn/cm^2, and the period of the leading wave is 2 to 3 min. The later waves have periods of 1 min or less, but the durations and amplitudes may not be accurately recorded in the sense that the frequency response of the instrument may not have been good enough.

At Geneva, elaborate microbarographic arrays were described, with a sensitivity of 0·1 dyn/cm^2, a noise-reducing device which would in exceptionally favourable meteorological conditions reduce the background perhaps to about 0·2 dyn/cm^2, and a flat frequency response over the range 0·025 to 2 c/s. This type of equipment gives true records of waves with periods between a few seconds and a minute. The wave trains recorded by such equipment from large explosions, or natural events, at large distances, last for a long time—sometimes more than one hour. There are hundreds of 'waves', and of course, a great deal of irregularity. In the wave train there are usually several sets of waves of big amplitude and fairly well-defined period. The average period of such waves varies with the size of the explosion.

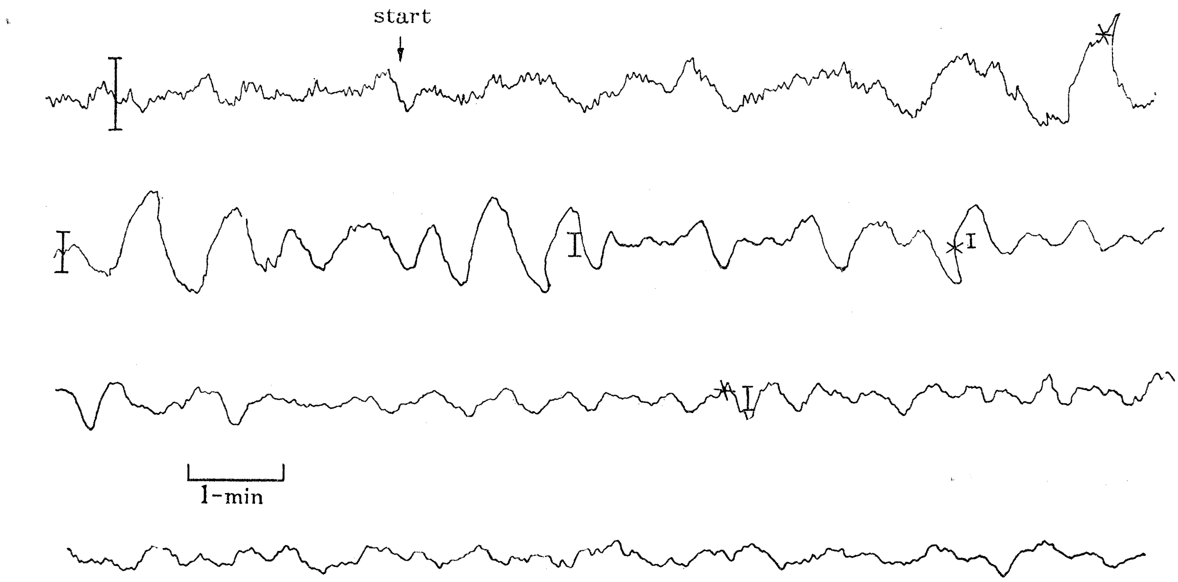
Smaller explosions also give a complicated wave train. However, as would be expected, the distances at which records can be obtained for smaller explosions are less than those at which records can be obtained for the larger explosions. The records obtained for the smaller explosions are therefore shorter in total duration because less time has elapsed between the explosion and the recording time, and the dispersion is correspondingly less. The amplitudes vary considerably with the meteorological conditions. If there are steady winds at 20 to 40 km height, with little wind shear, the signals in the downwind direction are much bigger than they are crosswind or upwind. For a 1 kT nuclear explosion on or near the ground, the greatest peak to trough amplitude in the waves at 1000 km has been observed as varying between 2 dyn/cm^2 and 32 dyn/cm^2. The most striking features of the wave trains are well-defined short-period waves, the period varying with the energy of the explosion. For example, waves with periods of about 10 s would be from explosions of a few kilotons, and waves with periods of more than a minute would be from explosions in the megaton range.

The experimental observations could be interpreted as follows. Explosions in the range 10^{20} to 10^{23} ergs at ground level give a train of gravity waves; and on the average, allowing for meteorological variations, the amplitudes and periods of the waves in the wave train observed at ground level are approximately proportional to the cube root of the energy of the explosion. Amplitudes, however, show considerable scatter.

Figure 9 shows some experimental records of explosions. The instrument which gave the record shown fourth did not have a frequency response capable of recording oscillations faster than of about 2 min duration. That is why the oscillations appear to be so much simpler than those shown in the other diagrams.



(i)



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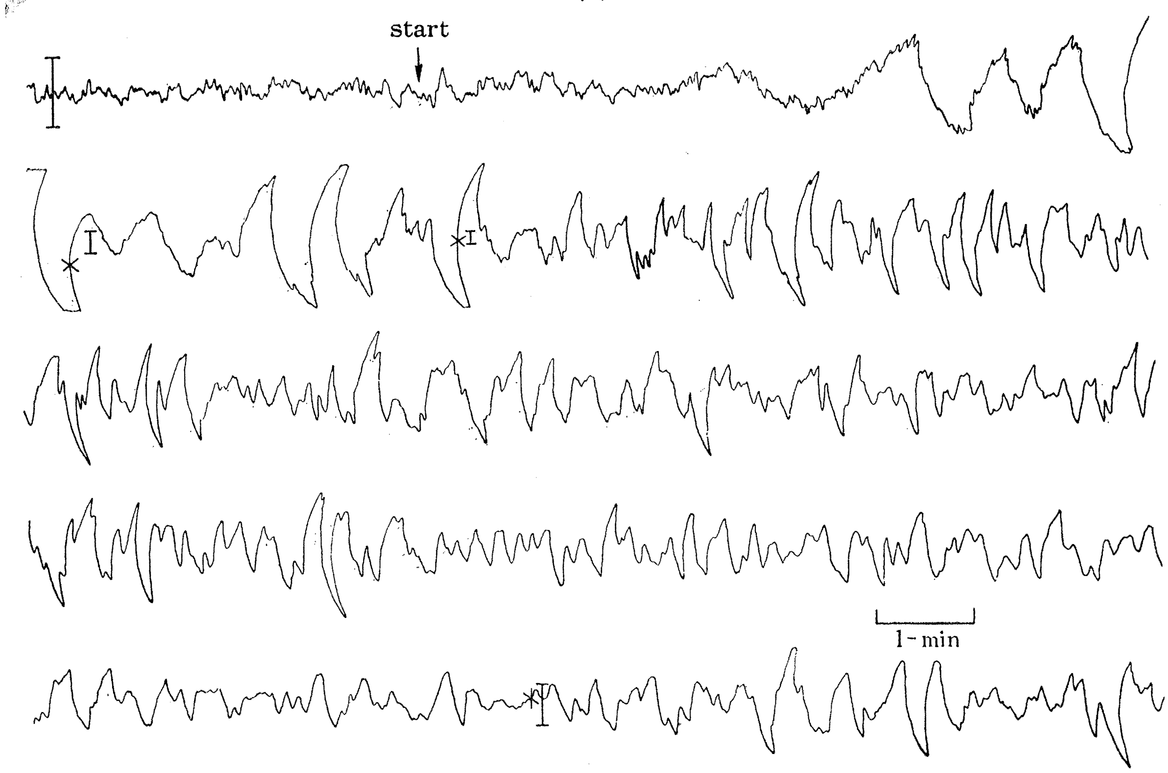

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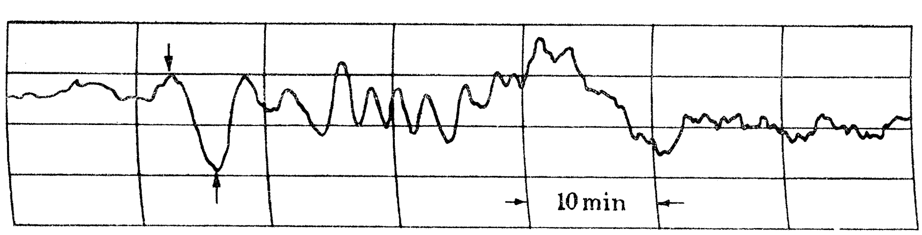

(iv)

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9. The blast wave

The present section makes estimates of the situation prevailing when the blast wave has travelled out to 75 km. The situation as calculated will be used in the following section as defining the starting conditions for the gravity wave train.

It is of course impossible to obtain an accurate analytical solution for the blast wave, and a numerical solution would require the extensive use of a large computer. However, we believe that the following simple treatment is adequate for our present purpose.

We could assume sound theory for extrapolating the pulse in the uniform atmosphere. Then the peak pressure and the peak velocity would fall away inversely as R, and the duration (or length) of the pulse would be constant. However, this approximation is not quite adequate. Because the disturbance is weakly finite, especially near the starting position, which might be taken to be where the overpressure is 1 Lb./in.^{2}, the peak pressure falls off faster than R^{-1}, and the pulse lengthens as it travels outwards. In the case of megaton explosions, the decay of PR is nearly 2, and the duration increases by nearly 2. It is our objective to calculate these factors with reasonable accuracy.

The pressure pulse at ground level from a large explosion at ground level can be estimated by the method of Brinkley & Kirkwood (1947). The shock overpressure P and the energy K per unit solid angle in the pulse are given by

$$\left.\begin{aligned} PR &= P_1\{\log_{10}(R/R_1)\}^{-\frac{1}{2}}, \\ K &= K_1\{\log_{10}(R/R_1)\}^{-\frac{1}{2}}, \end{aligned}\right\} \tag{69}$$

where P_1, K_1 and R_1 are constants. Thus, if we know P at two distances, we can calculate P at all distances, and find how K varies with distance.

U.S.A.E.C. (1957) gives for a 1 kT nuclear explosion at ground level

$$P = 3\ \text{Lb./in.}^2 \quad \text{when} \quad R = 0{\cdot}64\ \text{km},$$
$$P = 1\ \text{Lb./in.}^2 \quad \text{when} \quad R = 1{\cdot}38\ \text{km}.$$

Substituting in (69), we get $P_1 = 1{\cdot}145$ and $R_1 = 0{\cdot}286$ km. Therefore at $R = 75$ km, $P = 0{\cdot}00981$ Lb./in.^{2} and the energy in the blast at ground level per unit solid angle at 75 km is 53 % of the value at 1·38 km.

The expression for τ can be obtained from the volume integral giving the energy in the pulse. As the pulse passes radius R, the shape of the p–t curve has been assumed to be invariant relative to the scale factors P and τ. The approximation is now made that when the

Figure 9. The first diagram is a record obtained from an explosion of a few kilotons at a distance of a few thousand kilometres with favourable upper wind conditions. The second diagram is a record obtained from a large explosion at a distance of several thousand kilometres. The third diagram is a record obtained from a large explosion at more than 10000 km. The fourth is a record obtained in England from a large explosion in the Pacific. The time scale is the same in the first three records and the maximum amplitudes in the four records, peak to trough, are 2, 8, 20 and 70 dyn/cm^2. On the first three records, the sensitivity was changed once or more while the instrument was recording. The vertical lines give 2 dyn/cm^2 in the various parts of the record. The instrument with which the fourth record was obtained, while good for waves of period 1 to 2 min, was inadequate to record waves of shorter period. That is why the record appears much simpler than the others.

front is at R, the pulse inside R can be obtained by projecting backwards in radius each p and u value from the p–t and u–t curves at R, using a speed of travel c, the velocity of sound, and increasing the p and u values inversely as the radius r. This procedure for changing from a time variation in the pulse at a given radius to a radius variation in the pulse at a given time is asymptotically correct for sound theory. In the case of weakly finite blast pulses ($\leqslant 1$ Lb./in.^{2} overpressure), the error in the volume integral of the energy in the pulse will be small. The energy in the pulse when the front is at R is therefore $P^2R^2\tau \times$ some constant; and substituting the Brinkley–Kirkwood values for PR and K leads to an expression for τ:

$$\begin{aligned}\tau &= \tau_0(P_0/P_R)^2\,K_R/K_0\\ &= \tau_0\{\log_{10}(R/R_1)\}^{\frac{1}{2}}/\{\log_{10}(R_0/R_1)\}^{\frac{1}{2}},\end{aligned} \tag{70}$$

where τ_0, P_0, K_0 refer to some standard distance R_0.

For the 1 kT explosion on the ground, when $P = 1$ Lb./in.^{2}, the positive phase lasts for 0·53 s, and the negative phase is about three times as long. Taking this as the reference position, τ_0 is 2·12 s, and τ at 75 km is 1·88 times greater, or 3·98 s, at ground level.

Repeating the arithmetic for the case of a 1 MT explosion on the ground we estimate that at 75 km the duration at ground level is 30·5 s.

We must now consider the vertical variation of blast overpressure and total duration with height, at a distance of 75 km. First, we choose the size of the explosion and tabulate the durations and energies in the horizontal direction at a number of convenient radii, e.g. 2, 4, 8, 15, 20, 30, 40, 50, 60, 70, 75 km. We are going to do a step-by-step calculation for the blast which is travelling upwards at an angle θ to the vertical. Suppose that we have completed one step which has taken us to radius R_n and we now wish to extend our values to R_{n+1}. The average z value for this step is $\frac{1}{2}(R_n+R_{n+1})\sin\theta$ and the average air density is e^{-qz}, where q is $\gamma g/c^2$ or, say, $1{\cdot}4\times10^{-6}$. If $\delta\tau_n^0$ is the increase in duration at ground level between R_n and R_{n+1}, and if δK_n^0 is the loss of energy per unit solid angle between R_n and R_{n+1} at ground level, the corresponding quantities in the blast travelling at angle θ are

$$\left.\begin{aligned}\delta\tau_n^z &= Q\,\delta\tau_n^0,\\ \delta K_n^z &= Q^3\,\mathrm{e}^{-qz}\,\delta K_n^0.\end{aligned}\right\} \tag{71}$$

The meaning of Q is that it represents the enhancement of the shock-overpressure in terms of the local atmospheric pressure, due to the blast moving in thinner air.

We make a guess at a value for Q, and substitute back in the energy equation

$$\mathrm{e}^{-qz}\,Q^2\tau_{n+1}^z = \tau_{n+1}^0\,K_{n+1}^z/K_{n+1}^0. \tag{72}$$

There is no difficulty in quickly reaching the correct value of Q in each step.

By starting at a small initial radius for the first step, say 2 km, and neglecting the variation of τ around this hemisphere, one can compute the variation of blast pressure and duration along the direction θ, step by step.

The accuracy of the method depends on the assumption that the blast energy flows only radially. Moreover, it is necessary to verify that the 'lifting energy' against gravity in the blast pulse is small compared with the energy in the blast. It is in fact less than 1 % in the cases we have considered.

For 1 kT explosion at ground level, the duration of the pulse at ground level and distance 75 km is 3·98 s. At the same distance but at height 20 km the duration is 4·63 s. The corresponding durations for the 1 MT explosion on the ground are 30·5 and 38·6 s. The energies left in the blast at height 20 km are 86·3 and 80·2 % respectively of the values at ground level. The Q values at $R = 75$ km, $z = 20$ km are 3·34 and 3·09 respectively. If one made the approximation that the energy of the pulse and the duration were constant around the hemisphere $R = 75$ km, the Q value would be 3·94 at $z = 20$ km. Thus, the shock overpressure would be decreased by 3·94 relative to the value at ground level, but relative to the local atmospheric pressure would be enhanced by 3·94.

If we take the average duration of the blast wave from a large explosion at a distance 75 km, the average being taken between ground level and 40 km, as equal to the duration at the average height, 20 km, then the 1 kT explosion has a duration 4·63 s and the 1 MT explosion has a duration 38·6 s. The power-law dependence on energy is slightly less than the cube root, the power index being 0·31.

One can approximately scale the blast at any radius from one explosion of energy E_1 to another of energy E_2, where $j^3 = E_2/E_1$. The durations are increased by a factor somewhat less than j, and the amplitudes are increased by a factor somewhat greater than j. The errors in scaling from 1 kT to 1 MT are about 20 %.

10. Excitation of gravity waves in a three-layer atmosphere

In § 7, we have considered the gravity waves due to an explosive source at ground level in atmospheres with the stratosphere colder than the troposphere. This type of atmosphere has a simple spectrum of oscillations, extending only to a σ-value of approximately 0·05, or a period of 2 min.

Explosions of energy about 10^{23} ergs give a blast wave with a duration of the order of 1 min, and explosions of 10^{20} ergs give a blast with a duration of about 6 s. The source function Σ of (58) gives a fair representation of a blast wave provided T is put equal to one quarter of the blast duration. Thus $T = 15$ s for explosions of 10^{23} ergs, and $T = 1·5$ s for explosions of 10^{20} ergs. Since the maximum value of σ for the atmospheres with the colder stratospheres is at most only 0·05, the variation in Σ for the different σ values of the freely travelling waves is only a few parts per cent for explosions of 10^{20} ergs, and is of minor importance for explosions as large as 10^{23} ergs. Thus it is a good approximation to take Σ as constant for all allowed σ values. Consequently, the theory developed for model atmospheres with a colder stratosphere can only lead to the conclusion that the pressure oscillations at a distant point at ground level have amplitudes proportional to the energy of the explosion and periods independent of the energy of the explosion. It is a coincidence that the duration of the oscillations which we have calculated for various model atmospheres with colder stratospheres, being of the order of 1 or 2 min, happen to be about the same as those actually observed in the disturbances caused by very large explosions with an energy content of several megatons. However, the durations observed for kiloton explosions are of the order 10 s, and the theory fails to give such fast oscillations. One must conclude that model atmospheres with a colder stratosphere are unsatisfactory, at least for explosions in the kiloton range.

If one takes a model atmosphere with the top layer warmer than any of the lower layers, a much more elaborate spectrum of oscillations is permitted. High frequencies can propagate freely, and the higher the σ value, the more branches are permitted. A model atmosphere of this type appears to have a spectrum of oscillations with sufficient flexibility to give an explanation of the oscillations. One preferred model has three isothermal layers, with the middle layer acting as a 'sound channel'. Unfortunately, the complexity of the integrations over σ and k prevents a complete solution being obtained.

Table 9. Relative intensity function for branches $n=0$ and $n=1$ of three-layer atmosphere of table 5

$10^4\sigma^2$	$10^3 F(\sigma)$	$10^4\sigma^2$	$10^3 F(\sigma)$
branch $n=0$		branch $n=1$	
0·31	8·80	25·61	−49·09
1	8·25	26	−49·36
2	8·36	28	−51·88
4	9·08	30	−52·93
6	10·02	35	−53·00
8	11·05	40	−51·45
10	12·15	45	−49·29
15	15·08	50	−46·71
20	18·20	60	−41·09
30	24·90	80	−29·04
40	32·16	81	−28·44
50	39·94		
60	48·20		
80	66·07		
81	67·01		

The pressure disturbance at ground level is a simple extension of (61) and is

$$p_1 = \{\rho_{01} V/(2\pi r)^{\frac{1}{2}}\} \sum_n \int_{\sigma_{cn}}^{\infty} d\sigma\, \Sigma F_n(\sigma)\, (k_n)^{\frac{1}{2}} e^{i(\sigma t - kr + \frac{1}{4}\pi)}, \tag{73}$$

where σ_{cn} is the low-frequency cut-off for the branch n, and

$$F_n(\sigma) = 2\{(\sigma^2\tau_{01} + g\tau'_{01})/\sigma\tau_{01}\}\beta_n^2/\nu(\partial\beta_n^2/\partial k), \tag{74}$$

where

$$\left.\begin{aligned} \nu &= (2-\gamma)\, g/2c_1^2, \\ \beta_n^2 &= \{\sigma^2 - (\gamma-1)\, g^2/c_1^2\}\{1/c_1^2 - k^2/\sigma^2\} - \nu^2. \end{aligned}\right\} \tag{75}$$

The expression for $F_n(\sigma)$ reduces to

$$\begin{aligned} F_n(\sigma) &= -\sigma\beta_n^2/\nu k \quad (\sigma \geqslant \sigma_{cn}), \\ &= 0 \quad\quad\quad\;\; (\sigma < \sigma_{cn}). \end{aligned}$$

For the three-layer model, F_n increases with σ^2 for large σ; for the two-layer model, F_n tends to a constant for large σ. Table 9 gives values of $F_n(\sigma)$ for the first two branches of the three-layer model of table 5.

Only an elaborate investigation could cover all the regions of the integration over σ for the pressure disturbance. In each branch, the group velocity $d\sigma/dk$ has a stationary value, and, asymptotically for large σ, approaches the sound velocity in the middle channel. The complexity is not surprising. The pulse, as its spreads, is refracted and reflected at each interface, and in addition there is the complicated dispersion.

The method of stationary phase, in its simplest form, gives for the pressure disturbance at large distances at ground level

$$p_1 = \{\rho_{01} V/(ra \sin\theta)^{\frac{1}{2}}\} \sum_n F_n(\sigma)\, \Sigma (k_n/k''_n)^{\frac{1}{2}} \cos r(\sigma k'_n - k_n). \tag{76}$$

This approximation can be used near the start of a branch, but it fails completely at higher σ values, because k''_n becomes small. The high σ-components travel essentially as a non-dispersive pulse. Since the group velocity near the head of a branch is appreciably greater than the group velocity of the high σ values, the branch is spread out in time.

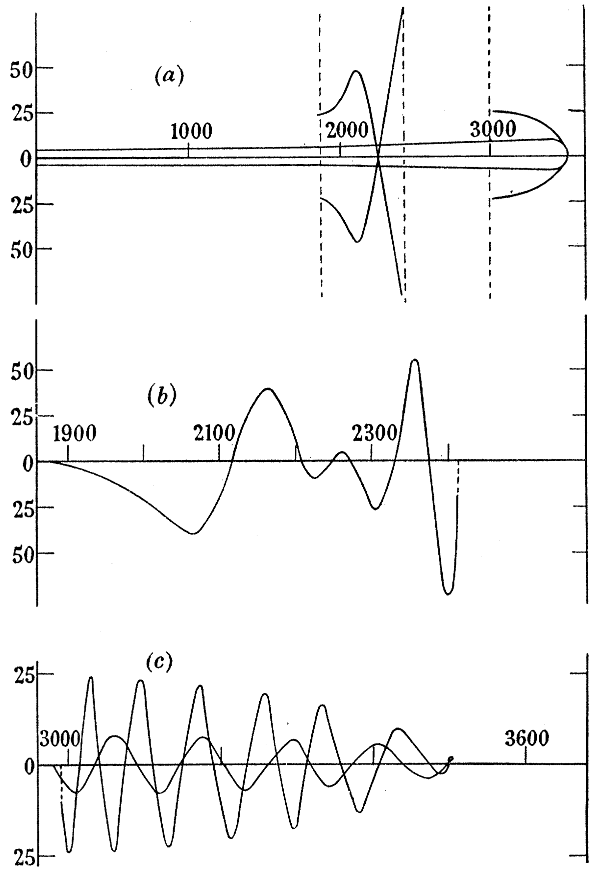

FIGURE 10. The pressure pulse on the ground in dynes cm^{-2} km^{-3} at distance 6400 km for waves of periods greater than 70 s of branches $n = 0$ and $n = 1$ of the three-layer model atmosphere of table 5, caused by an explosion of 5 MT, for which the blast duration is 63 s. (*a*) shows the envelope of the oscillations, which last for about one hour; (*b*) shows the oscillations due to branch $n = 0$, and (*c*) those due to branch $n = 1$, for waves with periods longer than 70 s. We have taken 1 cal of explosive energy as causing 8 cm^3 of air expansion.

Figure 10 gives the pressure oscillations in the branches $n = 0$ and $n = 1$ at 6400 km in the atmosphere described in tables 5 and 9, the Σ function being taken as that calculated for an explosion with a blast period 20π s, i.e. an explosion of 5 MT, including only waves with periods greater than 70 s. A comparison of figures 7*b*, 7*c* and 8 with the envelopes of the branches $n = 0$ and $n = 1$ of figure 10, shows that the amplitudes calculated for explosions of megaton scale using the three-layer model atmosphere are about two-thirds of those calculated for the 'mean European atmosphere' used by Scorer (model I of table 1), and are about the same as those calculated for a 'mean Pacific atmosphere' (model II of table 1). Using the three-layer model, the Siberian meteorite was about 10 MT, and the nuclear explosions of 1 March 1954 and 5 May 1954 were also about 10 MT.

The blast wave as the source

The complexity of the expression (76) for p_1, the disturbance at ground level, has arisen in part because the mathematics is attempting to represent a pulse which is reflected and refracted at the interfaces, as well as being dispersed. It may be helpful to study separately at least the refraction and total reflexion of the fundamental waves. In this way we obtain a clearer understanding of the phenomena.

The normal oscillations of the model atmospheres which we have considered are interference patterns of waves in $\varpi\, e^{-\nu z}$ of the form $e^{i(kr \pm \beta z - \sigma t)}$, where ν and β are given by (75).

These waves are travelling at an angle $\pm\theta$ to the vertical z-axis, where

$$\left.\begin{aligned} k = \kappa \sin\theta, \quad \beta = \kappa\cos\theta, \quad \kappa^2 = k^2+\beta^2, \\ \kappa^2 = (\sigma^2/c^2 - \gamma^2 g^2/4c^4)/\{1-(\gamma-1)\, g^2 \sin^2\theta/c^2\sigma^2\}. \end{aligned}\right\} \tag{77}$$

The group velocity is therefore

$$d\sigma/d\kappa = c(1-w)^{\frac{3}{2}}\,(1-\gamma^2 g^2/4c^2\sigma^2)^{\frac{1}{2}}/(1-2w+\gamma^2 g^2 w/4c^2\sigma^2), \tag{78}$$

where
$$w = (\gamma-1)\, g^2 \sin^2\theta/c^2\sigma^2.$$

Thus for waves with periods about 10 s travelling vertically upwards ($\theta = 0$), the group velocity is only 0·62 part in 1000 slower than the velocity of sound; and for waves of similar periods travelling upwards at 30° to the horizontal ($\theta = \frac{1}{3}\pi$), the group velocity is 0·50 part in 1000 slower than c. Horizontally, the group velocity is 0·115 part in 1000 slower than c. Since the wavelength is more than 3 km, it is clear that the dispersion in travelling 100 km is quite small. The main effect on the waves is an increase of amplitude with height due to the decrease in air density.

Consider the refraction of these waves at a horizontal interface of two isothermal layers, the upper being denoted by a suffix 2 and the lower by a suffix 1. The refractive index for a wave travelling from 1 into 2 is given by

$$\mu = \kappa_1/\kappa_2, \tag{79}$$

where κ_1 and κ_2 are obtained from (77) by using the appropriate velocity of sound. The parameters σ and k are of course the same in the two cases.

Strictly speaking, μ is a function of σ, and of the angle of incidence, but for the σ-values and temperature differences in which we are interested, it is a good approximation to neglect all but one of the terms in κ, in which case the familiar formula is obtained:

$$\mu = c_2/c_1. \tag{80}$$

If c_1 and c_2 differ by 10 %, then (80) gives the departure of μ from unity with 1 % accuracy for wave periods as long as 10 s.

If the upper layer is warmer than the lower layer, then a wave at an angle of incidence equal to $\sin^{-1}(1/\mu)$ gives a refracted wave which is horizontal. At greater angles of incidence there is total reflexion, and the amplitude in the upper layer dies off exponentially away from the interface.

If the upper layer is the colder, then the upward travelling wave is refracted towards the vertical, and total reflexion does not occur. However, a downward coming wave would be totally reflected at angles of incidence greater than the grazing angle. (This fact suggests than an explosion well above 40 km height would only weakly excite freely travelling gravity waves which could be recorded at great distances at ground level.)

Consider a stage where the blast wave has expanded sufficiently for it to have penetrated the uppermost warm layer. The shape is not quite hemispherical because of non-linear effects and because there has been refraction at the two interfaces, one at 10 km and the other at 40 km. As the blast wave moves further outwards, a time comes when the front is refracted horizontally at the upper interface. That part of the blast wave which at this stage is above the upper interface continues its upwards motion, and contributes nothing to the train of gravity waves at ground level at great distances. That part of the blast wave which is below the interface is totally reflected at the interface, and sets up a train of gravity waves which are 'progressive' in the horizontal direction and 'standing' in the vertical direction. There has also been, of course, some reflexion before total reflexion sets in.

The model we have chosen has a temperature −15 °C in the layer 0 to 10 km, −44 °C in the layer 10 to 40 km and 47 °C at 40 km and above. With these values, the blast wave separates into its two parts when the horizontal radius of the blast is about 75 km.

Consider figure 11. The outermost circular arc represents the front of the blast wave when its radius is 75 km. The other circular arcs give the limits of the positive phase and the negative phase. The horizontal line represents the nodal planes of the σ harmonic oscillation of the branch n of the model atmosphere. For simplicity we shall take the blast wave as a single sinusoidal oscillation with an 'effective' wave length L. The value we take for L is simply the velocity of sound c times the duration τ of the blast wave. (Figure 6 shows that this is the wavelength most excited by the blast.)

There is approximate 'matching' between the blast wave and the oscillation of the nth branch in the region of a point P provided

$$r/z \sim k/\beta \sim \tan\theta, \quad 2\pi/k \sim L \sim 2\pi c/\sigma.$$

Moreover, we have approximately that

$$\beta = (2n+1)\,\pi/2(z_3-z_2),$$

where $$z_2 = 10\,\text{km} \quad \text{and} \quad z_3 = 40\,\text{km}.$$

Thus, that part of the blast wave at height z is exciting σ values approximately equal to $2\pi c/L$ in the nth branch, where

$$z = (2n+1)\,rL/4(z_3-z_2).$$

Since the maximum value of z is 40 km, we find on substituting the numerical values that the maximum value of n is given by

$$(2n+1)\,L = 64\,\text{km}.$$

A blast wave with an effective length of 2 km excites all n values up to about 16; while a blast wave with an effective length of 20 km excites only branches up to $n = 1$, i.e. it excites only $n = 0$ and $n = 1$.

We have shown in § 9 that an explosion of 1 kT gives a blast wave with a duration of 4·63 s, i.e. an effective L value of about 1·5 km. A 1 kT explosion therefore excites about twenty branches. An explosion of 1 MT has an effective length about ten times greater, and therefore substantially excites only the first two branches.

Formally, of course, any explosion will excite every permitted value of every branch. The oscillations at ground level at a distant point due to the excitation of the nth branch will therefore be of the form

$$(\xi_n/r)^{\frac{1}{2}} \int_0^\infty A_{n\sigma}\, e^{i\{\sigma t - k_n(r-\xi_n)+\frac{1}{4}\pi\}}\, d\sigma,$$

where $A_{n\sigma}$ is the excitation function for the branch, k_n is a function of σ as calculated for the branch and ξ_n is the horizontal co-ordinate of that region of the blast wave which excites the nth branch at the starting position, i.e. point P of figure 11.

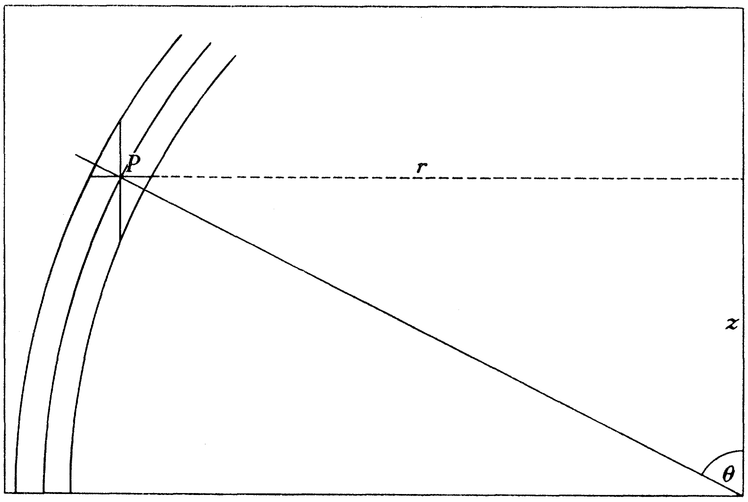

FIGURE 11. The blast wave in the vicinity of the point P is exciting the nth branch of the freely travelling gravity waves because there is a good match near P in both the horizontal and vertical 'wavelengths'.

Using the method of stationary phase, in a way similar to that used in obtaining (61), we get that the oscillations at a distant point r are given by

$$\delta p_n = S_n(\sigma) \cos\{(r-\xi_n)(k_n - \sigma k_n')\},$$

where time is calculated from $$t = (r-\xi_n)\, k_n',$$

S_n is a weighting function dependent on the size of the explosion, and ξ_n is a starting radius varying between 75 and 65 km, according to the size of the explosion and the branch which is excited.

The weighting function has a blunt maximum for σ values equal to $2\pi c/L$, where L is the length of the blast wave as it would be in an isotropic atmosphere at a distance of a few tens of kilometres. In other words, the greatest waves in the disturbance should have an average period roughly equal to the duration of the blast wave at a distance of a few tens of kilometres in an isotropic atmosphere.

The pressure disturbance at r will be the sum of the pressure oscillations in each branch. The times at which each branch starts and ends at point r are independent of the size of the explosion, but when the sum over all branches is taken, the pressure disturbance at r is quite different for explosions of widely different energies. At distances of 1000 km or more, the pressure disturbance in the simple model atmosphere we have considered is extremely complicated. When the effects of atmospheric wind structure and turbulence are imagined to be superimposed, the pressure oscillations will have a truly formidable complexity, comparable with that in the experimental records of figure 9. However, as we have shown, the average period of the largest waves gives an indication of the size of the explosion; and in the case of the explosions of 10^{15} cal or more, the amplitudes of the biggest waves usually will give a confirming value of the size.

Perhaps our most interesting conclusion is that the Siberian meteorite had an energy of $(10 \pm 5) \times 10^{15}$ cal, or 10 ± 5 MT, and was close in energy to the largest of the man-made nuclear explosions.

References

Astapowitsch, I. S. 1934 *Quart. J. R. Met. Soc.* **60**, 493.
Brinkley, S. R. & Kirkwood, J. G. 1947 *Phys. Rev.* **71**, 606.
Green, H. L. & Lane, W. R. 1957 *Particulate clouds, dusts, smokes, and mists.* London: Spon's General and Industrial Chemistry Series.
Havens, R. J., Koll, R. & La Gow, H. E. 1952 *J. Geophys. Res.* **57**, 59.
Kulik, L. A. 1927 *C.R. Acad. Sci. U.R.S.S.* **23**, 399.
Lamb, H. 1932 *Hydrodynamics*. Cambridge University Press.
Meissner, E. 1921 *Vjscht. Natur. Ges. Zurich* **66**, 181.
Pekeris, C. L. 1937 *Proc. Roy. Soc.* A, **158**, 650.
Pekeris, C. L. 1939 *Proc. Roy. Soc.* A, **171**, 435.
Pekeris, C. L. 1948 *Phys. Rev.* **73**, 145.
Scorer, R. S. 1950 *Proc. Roy. Soc.* A, **201**, 137.
U.S.A.E.C. 1957 *The effects of nuclear weapons.*
Whipple, F. J. W. 1930 *Quart. J. R. Met. Soc.* **56**, 287.
Whipple, F. J. W. 1934 *Quart. J. R. Met. Soc.* **60**, 505.
Wilkes, M. V. & Weekes, K. 1947 *Proc. Roy. Soc.* A, **192**, 80.
Yamamoto, R. 1954 *Met. Notes Kyoto Univ.* **2**, 120.
Yamamoto, R. 1956 *J. Met. Soc. Japan* (2), **34**, 322.
Yamamoto, R. 1957 *J. Met. Soc. Japan* (2), **35**, 281.